Mean-Field Backward Stochastic Volterra Integral Equations*

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Abstract

Mean-field backward stochastic Volterra integral equations (MF-BSVIEs, for short) are introduced and studied. Well-posedness of MF-BSVIEs in the sense of introduced adapted M-solutions is established. Two duality principles between linear mean-field (forward) stochastic Volterra integral equations (MF-FSVIEs, for short) and MF-BSVIEs are obtained. Several comparison theorems for MF-FSVIEs and MF-BSVIEs are proved. A Pontryagin's type maximum principle is established for an optimal control of MF-FSVIEs.

Keywords. Mean-field stochastic Volterra integral equation, mean-field backward stochastic Volterra integral equation, duality principle, comparison theorem, maximum principle.

AMS Mathematics subject classification. 60H20, 93E20, 35Q83.

1 Introduction.

Throughout this paper, we let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $W(\cdot)$ is defined with $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ being its natural filtration augmented by all the $\mathbb{P}$-null sets. Let us begin with the following stochastic differential equation (SDE, for short) in $\mathbb{R}$:

$$\begin{cases} dX(t) = b(X(t), \mu(t))dt + dW(t), \qquad t \in [0, T], \\ X(0) = x, \end{cases} \tag{1.1}$$

where

$$\begin{aligned} b(X(t), \mu(t)) &= \int_\Omega b(X(t,\omega), X(t;\omega'))\mathbb{P}(d\omega') \\ &\equiv \int_{\mathbb{R}} b(\xi, y)\mu(t; dy)\Big|_{\xi=X(t)} \equiv \mathbb{E}[b(\xi, X(t))]\Big|_{\xi=X(t)}, \end{aligned} \tag{1.2}$$

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where $b : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a (locally) bounded Borel measurable function and $\mu(t;\cdot)$ is the probability distribution of the unknown process $X(t)$:

$$\mu(t;A) = \mathbb{P}(X(t)\in A), \qquad \forall A\in\mathcal{B}(\mathbb{R}). \tag{1.3}$$

Here $\mathcal{B}(\mathbb{R}^n)$ is the Borel σ-field of $\mathbb{R}^n$ ($n\ge 1$). Equation (1.1) is called a *McKean–Vlasov* SDE. Such an equation was suggested by Kac [20] as a stochastic toy model for the Vlasov kinetic equation of plasma and the study of which was initiated by McKean [25]. Since then, many authors made contributions on McKean–Vlasov type SDEs and applications, see, for examples, Dawson [14], Dawson–Gärtner [15], Gártner [16], Scheutzow [32], Sznitman [33], Graham [17], Chan [11], Chiang [12], Ahmed–Ding [2]. In recent years, related topics and problems have attracted more and more attentions, see, for examples, Veretennikov [38], Huang–Malhamé–Caines [19], Ahmed [1], Mahmudov–McKibben [24], Lasry–Lions [22], Borkar–Kumar [7], Crisan–Xiong [13], Kotelenez–Kurtz [21], Park–Balasubramaniam–Kang [27], Andersson–Djehiche [4], Meyer-Brandis–Oksendal–Zhou [26], and so on.

Inspired by (1.1), one can consider the following more general SDE:

$$\begin{cases} dX(t) = b(t,X(t),\mathbb{E}[\theta^b(t,\xi,X(t))]_{\xi=X(t)})dt \\ \qquad\qquad\qquad +\sigma(t,X(t),\mathbb{E}[\theta^\sigma(t,\xi,X(t))]_{\xi=X(t)})dW(t), \qquad t\in[0,T], \\ X(0)=x. \end{cases} \tag{1.4}$$

where θ^b and θ^σ are some suitable maps. We call the above a *mean-field* (forward) stochastic differential equation (MF-FSDE, for short). From (1.2) and (1.4), we see that (1.1) is a special case of (1.4). Note also that (1.4) is an extension of classical Itô type SDEs. Due to the dependence of b and σ on $\mathbb{E}[\theta^b(t,\xi,X(t))]_{\xi=X(t)}$ and $\mathbb{E}[\theta^\sigma(t,\xi,X(t))]_{\xi=X(t)}$, respectively, MF-FSDE (1.4) is *nonlocal* with respect to the event $\omega\in\Omega$.

It is easy to see that the equivalent integral form of (1.4) is as follows:

$$\begin{aligned} X(t) = x &+ \int_0^t b(s,X(s),\mathbb{E}[\theta^b(s,\xi,X(s))]_{\xi=X(s)})ds \\ &+ \int_0^t \sigma(s,X(s),\mathbb{E}[\theta^\sigma(s,\xi,X(s))]_{\xi=X(s)})dW(s), \qquad t\in[0,T]. \end{aligned} \tag{1.5}$$

This suggests a natural extension of the above to the following:

$$\begin{aligned} X(t) = \varphi(t) &+ \int_0^t b(t,s,X(s),\mathbb{E}[\theta^b(t,s,\xi,X(s))]_{\xi=X(s)})ds \\ &+ \int_0^t \sigma(t,s,X(s),\mathbb{E}[\theta^\sigma(t,s,\xi,X(s))]_{\xi=X(s)})dW(s), \qquad t\ge 0. \end{aligned} \tag{1.6}$$

We call the above a mean-field (forward) stochastic Volterra integral equation (MF-FSVIE, for short). It is worthy of pointing out that when the drift b and diffusion σ in (1.6) are independent of the nonlocal terms $\mathbb{E}[\theta^b(t,s,\xi,X(s))]_{\xi=X(s)}$ and $\mathbb{E}[\theta^\sigma(t,s,\xi,X(s))]_{\xi=X(s)}$, respectively, (1.6) is reduced to a so-called (forward) stochastic Volterra integral equation (FSVIEs, for short):

$$X(t) = \varphi(t) + \int_0^t b(t,s,X(s))ds + \int_0^t \sigma(t,s,X(s))dW(s), \qquad t\ge 0. \tag{1.7}$$

Such kind of equations have been studied by a number of researchers, see, for examples, Berger–Mizel [6], Protter [30], Pardoux–Protter [29], Tudor [34], Zhang [43], and so on. Needless to say, the theory for (1.6) is very rich and have a great application potential in various areas.

On the other hand, a general (nonlinear) backward stochastic differential equation (BSDE, for short) introduced in Pardoux–Peng [28] is equivalent to the following:

$$Y(t)=\xi+\int_t^T g(s,Y(s),Z(s))ds-\int_t^T Z(s)dW(s),\qquad t\in[0,T]. \tag{1.8}$$

Extending the above, the following general stochastic integral equation was introduced and studied in Yong [39, 40, 41]:

$$Y(t)=\psi(t)+\int_t^T g(t,s,Y(s),Z(t,s),Z(s,t))ds-\int_t^T Z(t,s)dW(s),\qquad t\in[0,T]. \tag{1.9}$$

Such an equation is called a backward stochastic Volterra integral equation (BSVIE, for short). A special case of (1.9) with $g(\cdot)$ independent of $Z(s,t)$ and $\psi(t)\equiv\xi$ was studied by Lin [23] and Aman–N'zi [3] a little earlier. Some relevant studies of (1.9) can be found in Wang–Zhang [37], Wang–Shi [36], Ren [31], and Anh–Grecksch–Yong [5]. Inspired by BSVIEs, it is very natural for us to introduce the following stochastic integral equation:

$$\begin{aligned}Y(t)=\psi(t)+\int_t^T g(t,s,Y(s),Z(t,s),Z(s,t),\Gamma(t,s,Y(s),Z(t,s),Z(s,t)))ds\\ -\int_t^T Z(t,s)dW(s),\qquad t\in[0,T],\end{aligned} \tag{1.10}$$

where $(Y(\cdot),Z(\cdot\,,\cdot))$ is the pair of unknown processes, $\psi(\cdot)$ is a given *free term* which is $\mathcal{F}_T$-measurable (not necessarily $\mathbb{F}$-adapted), $g(\cdot)$ is a given mapping, called the *generator*, and

$$\Gamma(t,s,Y,Z,\widehat Z)=\mathbb{E}\Big[\theta(t,s,y,z,\hat z,Y,Z,\widehat Z)\Big]_{(y,z,\hat z)=(Y,Z,\widehat Z)} \tag{1.11}$$

with $(Y,Z,\widehat Z)$ being some random variables, for some mapping $\theta(\cdot)$ (see the next section for precise meaning of the above). We call (1.10) a *mean-field backward stochastic Volterra integral equation* (MF-BSVIE, for short). Relevant to the current paper, let us mention that in Buckdahn–Djehiche–Li–Peng [9], mean-field backward stochastic differential equations (MF-BSDEs, for short) were introduced and in Buckdahn–Li–Peng [10] a class of nonlocal PDEs are studied with the help of an MF-BSDE and a McKean-Vlasov forward equation.

We see that MF-BSVIE (1.10) not only includes MF-BSDEs (which, of course, also includes standard BSDEs) introduced in [9, 10], but also generalizes BSVIEs studied in [39, 41, 36], etc. in a natural way. Besides, investigating MF-BSVIEs allows us to meet the need in the study of optimal control for MF-FSVIEs. As a matter of fact, in the statement of Pontryagin type maximum principle for optimal control of a forward (deterministic or stochastic) control system, the adjoint equation of variational state equation is a corresponding (deterministic or stochastic) backward system, see [42] for the case of classical optimal control problems, [4, 8, 26] for the case of MF-FSDEs, and [39, 41] for the case of FSVIEs. When the state equation is an MF-FSVIE, the adjoint equation will naturally be an MF-BSVIE. Hence the study of well-posedness for MF-BSVIEs is not avoidable when we want to study optimal control problems for MF-BSVIEs.

The novelty of this paper mainly contains the following: First, well-posedness of general MF-BSVIEs will be established. In doing that, we discover that the growth of the generator and the nonlocal term with respect to $Z(s,t)$ plays a crucial role; a better understanding of which enables us to have found a neat way of treating term $Z(s,t)$. Even for BSVIEs, our new method will significantly simplify the proof of well-posedness of the equation (comparing with [41]). Second, we establish two slightly different duality principles, one starts from linear MF-FSVIEs, and the other starts from linear MF-BSVIEs. We found that "*Twice adjoint of a linear MF-FSVIE is itself*", whereas, "*Twice adjoint of a linear MF-BSVIE is not necessarily itself*". Third, some comparison theorems will be established for MF-FSVIEs and MF-BSVIEs. It turns out that the situation is surprisingly different from the differential equation cases. Some mistakes found in [39, 40] will be corrected. Finally, as an application of the duality principle for MF-FSVIEs, we establish a Pontryagin type maximum principle for an optimal control problem of MF-FSVIEs.

The rest of the paper is organized as follows. Section 2 is devoted to present some preliminary results. In Section 3, we prove the existence and uniqueness of adapted M-solutions to MF-BSVIE (1.10). In Section 4 we obtain duality principles. Comparison theorems will be presented in Section 5. In Section 6, we deduce a maximum principle of optimal controls for MF-FSVIEs.

2 Preliminary Results.

In this section, we will make some preliminaries.

2.1 Formulation of MF-BSVIEs.

Let us first introduce some spaces. For $H=\mathbb{R}^n$, etc., and $p>1$, $t\in[0,T]$, let

$$\begin{aligned}
&L^p(0,T;H)=\Big\{x:[0,T]\to H\;\Big|\;\int_0^T|x(s)|^p ds<\infty\Big\},\\
&L^p_{\mathcal{F}_t}(\Omega;H)=\Big\{\xi:\Omega\to H\;\Big|\;\xi\text{ is }\mathcal{F}_t\text{-measurable, }\mathbb{E}|\xi|^p<\infty\Big\},\\
&L^p_{\mathcal{F}_t}(0,T;H)=\Big\{X:[0,T]\times\Omega\to H\;\Big|\;X(\cdot)\text{ is }\mathcal{F}_t\text{-measurable, }\mathbb{E}\int_0^T|X(s)|^p ds<\infty\Big\},\\
&L^p_{\mathbb{F}}(0,T;H)=\Big\{X:[0,T]\times\Omega\to H\;\Big|\;X(\cdot)\text{ is }\mathbb{F}\text{-adapted, }\mathbb{E}\int_0^T|X(s)|^p ds<\infty\Big\},\\
&L^p_{\mathbb{F}}(\Omega;L^2(0,T;H))=\Big\{X:[0,T]\times\Omega\to H\;\Big|\;X(\cdot)\text{ is }\mathbb{F}\text{-adapted, }\mathbb{E}\Big(\int_0^T|X(s)|^2 ds\Big)^{\frac{p}{2}}<\infty\Big\}.
\end{aligned}$$

Also, let (with $q\ge 1$)

$$\begin{aligned}
L^p(0,T;L^q_{\mathbb{F}}(0,T;H))=\Big\{&Z:[0,T]^2\times\Omega\to H\;\Big|\;Z(t,\cdot)\text{ is }\mathbb{F}\text{-adapted for almost all }t\in[0,T],\\
&\mathbb{E}\int_0^T\Big(\int_0^T|Z(t,s)|^q ds\Big)^{\frac{p}{q}}dt<\infty\Big\},\\
C^p_{\mathbb{F}}([0,T];H)=\Big\{&X:[0,T]\times\Omega\to H\;\Big|\;X(\cdot)\text{ is }\mathbb{F}\text{-adapted, }t\mapsto X(t)\text{ is continuous}\\
&\text{from }[0,T]\text{ to }L^p_{\mathcal{F}_T}(\Omega;H),\;\sup_{t\in[0,T]}\mathbb{E}[|X(t)|^p]<\infty\Big\}.
\end{aligned}$$

We denote

$$\begin{aligned}&\mathcal{H}^p[0,T]=L^p_{\mathbb{F}}(0,T;H)\times L^p(0,T;L^2_{\mathbb{F}}(0,T;H)),\\&\mathbb{H}^p[0,T]=C^p_{\mathbb{F}}([0,T];H)\times L^p_{\mathbb{F}}(\Omega;L^2(0,T;H)).\end{aligned}$$

Next, let $(\Omega^2,\mathcal{F}^2,\mathbb{P}^2)=(\Omega\times\Omega,\mathcal{F}\otimes\mathcal{F},\mathbb{P}\otimes\mathbb{P})$ be the completion of the product probability space of the original $(\Omega,\mathcal{F},\mathbb{P})$ with itself, where we define the filtration as $\mathbb{F}^2=\{\mathcal{F}_t\otimes\mathcal{F}_t,\ t\in[0,T]\}$ with $\mathcal{F}_t\otimes\mathcal{F}_t$ being the completion of $\mathcal{F}_t\times\mathcal{F}_t$. It is worthy of noting that any random variable $\xi=\xi(\omega)$ defined on Ω can be extended naturally to Ω^2 as $\xi'(\omega,\omega')=\xi(\omega)$, with $(\omega,\omega')\in\Omega^2$. Similar to the above, we define

$$L^1(\Omega^2,\mathcal{F}^2,\mathbb{P}^2;H)=\Big\{\xi:\Omega^2\to H\ \Big|\ \xi\text{ is }\mathcal{F}^2\text{-measurable, }\mathbb{E}^2|\xi|\equiv\int_{\Omega^2}|\xi(\omega',\omega)|\mathbb{P}(d\omega')\mathbb{P}(d\omega)<\infty\Big\}.$$

For any $\eta\in L^1(\Omega^2,\mathcal{F}^2,\mathbb{P}^2;H)$, we denote

$$\mathbb{E}'\eta(\omega,\cdot)=\int_\Omega\eta(\omega,\omega')\mathbb{P}(d\omega')\in L^1(\Omega,\mathcal{F},\mathbb{P}).$$

Note that if $\eta(\omega,\omega')=\eta(\omega')$, then

$$\mathbb{E}'\eta=\int_\Omega\eta(\omega')\mathbb{P}(d\omega')=\int_\Omega\eta(\omega)\mathbb{P}(d\omega)=\mathbb{E}\eta.$$

In what follows, $\mathbb{E}'$ will be used when we need to distinguish ω' from ω, which is the case when both ω and ω' appear at the same time. Finally, we denote

$$\Delta=\Big\{(t,s)\in[0,T]^2\ \Big|\ t\le s\Big\},\quad \Delta^*=\Big\{(t,s)\in[0,T]^2\ \Big|\ t\ge s\Big\}\equiv\overline{\Delta^c}.$$

Let

$$g:\Delta\times\Omega\times\mathbb{R}^{3n}\times\mathbb{R}^m\to\mathbb{R}^n,\qquad \theta:\Delta\times\Omega^2\times\mathbb{R}^{6n}\to\mathbb{R}^m,\tag{2.1}$$

be some suitable maps (see below for precise conditions) and define

$$\begin{aligned}&\Gamma(t,s,Y,Z,\widehat Z)=\mathbb{E}'\Big[\theta(t,s,y,z,\widehat z,Y,Z,\widehat Z)\Big]_{(y,z,\hat z)=(Y,Z,\widehat Z)}\\&=\int_\Omega\theta(t,s,\omega,\omega',Y(\omega),Z(\omega),\widehat Z(\omega),Y(\omega'),Z(\omega'),\widehat Z(\omega'))\mathbb{P}(d\omega'),\end{aligned}\tag{2.2}$$

for all reasonable random variables $(Y,Z,\widehat Z)$. This gives the precise meaning of (1.11). Hereafter, when we talk about MF-BSVIE (1.10), the mapping Γ is defined by (2.2). With such a mapping, we have

$$\begin{aligned}&\Gamma(t,s,Y(s),Z(t,s),Z(s,t))\equiv\Gamma(t,s,\omega,Y(s,\omega),Z(t,s,\omega),Z(s,t,\omega))\\&=\int_\Omega\theta(t,s,\omega,\omega',Y(s,\omega),Z(t,s,\omega),Z(s,t,\omega),Y(s,\omega'),Z(t,s,\omega'),Z(s,t,\omega'))\mathbb{P}(d\omega').\end{aligned}$$

Clearly, the operator Γ is *nonlocal* in the sense that the value $\Gamma(t,s,\omega,Y(s,\omega),Z(t,s,\omega),Z(s,t,\omega))$ of $\Gamma(t,s,Y(s),Z(t,s),Z(s,t))$ at ω depends on the whole set

$$\{(Y(s,\omega'),Z(t,s,\omega'),Z(s,t,\omega'))\mid\omega'\in\Omega\},$$

not just on $(Y(s,\omega),Z(t,s,\omega),Z(s,t,\omega))$. To get some feeling about such an operator, let us look at a simple but nontrivial special case.

Example 2.1. Let

$$\begin{aligned}\theta(t,s,\omega,\omega',y,z,\hat z,y',z',\hat z')&=\theta_0(t,s,\omega)+A_0(t,s,\omega)y+B_0(t,s,\omega)z+C_0(t,s,\omega)\hat z\\&\quad+A_1(t,s,\omega,\omega')y'+B_1(t,s,\omega,\omega')z'+C_1(t,s,\omega,\omega')\hat z'.\end{aligned}$$

We should carefully distinguish ω' and ω in the above. Then (suppressing ω)

$$\begin{aligned}\Gamma(t,s,Y(s),Z(t,s),Z(s,t))&=\theta_0(t,s)+A_0(t,s)Y(s)+B_0(t,s)Z(t,s)+C_0(t,s)Z(s,t)\\&\quad+\mathbb{E}'[A_1(t,s)Y(s)]+\mathbb{E}'[B_1(t,s)Z(t,s)]+\mathbb{E}'[C_1(t,s)Z(s,t)],\end{aligned}$$

where, for example,

$$\mathbb{E}'[B_1(t,s)Z(t,s)]=\int_\Omega B_1(t,s,\omega,\omega')Z(t,s,\omega')\mathbb{P}(d\omega').$$

For such a case, $(Y(\cdot),Z(\cdot\,,\cdot))\mapsto\Gamma(\cdot\,,\cdot\,,Y(\cdot),Z(\cdot\,,\cdot),Z(\cdot\,,\cdot))$ is affine.

Having some feeling about the operator Γ from the above, let us look at some useful properties of the operator Γ in general. To this end, we make the following assumption.

(H0)$_q$ The map $\theta:\Delta\times\Omega^2\times\mathbb{R}^{6n}\to\mathbb{R}^m$ is measurable and for all $(t,y,z,\hat z,y',z',\hat z')\in[0,T]\times\mathbb{R}^{6n}$, the map $(s,\omega,\omega')\mapsto\theta(t,s,\omega,\omega',y,z,\hat z,y',z',\hat z')$ is $\mathbb{F}^2$-progressively measurable on $[t,T]$. Moreover, there exist constants $L>0$ and $q\ge 2$ such that

$$\begin{aligned}&|\theta(t,s,\omega,\omega',y_1,z_1,\hat z_1,y_1',z_1',\hat z_1')-\theta(t,s,\omega,\omega',y_2,z_2,\hat z_2,y_2',z_2',\hat z_2')|\\&\le L\Big(|y_1-y_2|+|z_1-z_2|+|\hat z_1-\hat z_2|+|y_1'-y_2'|+|z_1'-z_2'|+|\hat z_1'-\hat z_2'|\Big),\\&\qquad\qquad\forall(t,s,\omega,\omega')\in\Delta\times\Omega^2,\ (y_i,z_i,\hat z_i,y_i',z_i',\hat z_i')\in\mathbb{R}^{6n},i=1,2,\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}&|\theta(t,s,\omega,\omega',y,z,\hat z,y',z',\hat z')|\le L\Big(1+|y|+|z|+|\hat z|^{\frac2q}+|y'|+|z'|+|\hat z'|^{\frac2q}\Big),\\&\qquad\qquad\forall(t,s,\omega',\omega)\in\Delta\times\Omega^2,\ (y,z,\hat z,y',z',\hat z')\in\mathbb{R}^{6n}.\end{aligned}\tag{2.4}$$

In the above, we may replace constant L by some function $L(t,s)$ with certain integrability (similar to [41]). However, for the simplicity of presentation, we prefer to take a constant L. Also, we note that $(\hat z,\hat z')\mapsto\theta(t,s,\omega,\omega',y,z,\hat z,y',z',\hat z')$ is assumed to grow no more than $|\hat z|^{\frac2q}+|\hat z'|^{\frac2q}$. If $q=2$, then the growth is linear and if $q>2$, the growth is sublinearly. This condition is very subtle in showing that the solution $(Y(\cdot),Z(\cdot\,,\cdot))$ of MF-BSVIE belongs in $\mathcal{H}^q[0,T]$. We would like to mention that $(\mathrm{H0})_\infty$ is understood as that (2.4) is replaced by the following

$$\begin{aligned}&|\theta(t,s,\omega,\omega',y,z,\hat z,y',z',\hat z')|\le L\Big(1+|y|+|z|+|y'|+|z'|\Big),\\&\qquad\qquad\forall(t,s,\omega',\omega)\in\Delta\times\Omega^2,\ (y,z,\hat z,y',z',\hat z')\in\mathbb{R}^{6n}.\end{aligned}\tag{2.5}$$

Under $(\mathrm{H0})_q$, for any $(Y(\cdot),Z(\cdot\,,\cdot))\in\mathcal{H}^q[0,T]$, we see that for each $t\in[0,T]$, the map

$$\begin{aligned}&(s,\omega)\mapsto\Gamma(t,s,\omega,Y(s),Z(t,s),Z(s,t))\\&\equiv\int_\Omega\theta(t,s,\omega,\omega',Y(s,\omega),Z(t,s,\omega),Z(s,t,\omega),Y(s,\omega'),Z(t,s,\omega'),Z(s,t,\omega'))\mathbb{P}(d\omega')\end{aligned}$$

is $\mathbb{F}$-progressively measurable on $[t,T]$. Also,

$$
\begin{aligned}
&|\theta(t,s,\omega,\omega',Y(s,\omega),Z(t,s,\omega),Z(s,t,\omega),y,z,\hat z)|\\
&\le L\Big(1+|Y(s,\omega)|+|Z(t,s,\omega)|+|Z(s,t,\omega)|^{\frac{2}{q}}+|y|+|z|+|\hat z|^{\frac{2}{q}}\Big).
\end{aligned}
\tag{2.6}
$$

Consequently,

$$
\begin{aligned}
&|\Gamma(t,s,Y(s),Z(t,s),Z(s,t))|\\
&\le L\Big(1+|Y(s)|+|Z(t,s)|+|Z(s,t)|^{\frac{2}{q}}+\mathbb{E}|Y(s)|+\mathbb{E}|Z(t,s)|+\mathbb{E}|Z(s,t)|^{\frac{2}{q}}\Big).
\end{aligned}
\tag{2.7}
$$

Likewise, for any $(Y_1(\cdot),Z_1(\cdot\,,\cdot)),(Y_2(\cdot),Z_2(\cdot\,,\cdot))\in\mathcal{H}^q[0,T]$, we have

$$
\begin{aligned}
&|\Gamma(t,s,Y_1(s),Z_1(t,s),Z_1(s,t))-\Gamma(t,s,Y_2(s),Z_2(t,s),Z_2(s,t))|\\
&\le L\Big(|Y_1(s)-Y_2(s)|+|Z_1(t,s)-Z_2(t,s)|+|Z_1(s,t)-Z_2(s,t)|\\
&\qquad+\mathbb{E}|Y_1(s)-Y_2(s)|+\mathbb{E}|Z_1(t,s)-Z_2(t,s)|+\mathbb{E}|Z_1(s,t)-Z_2(s,t)|\Big).
\end{aligned}
\tag{2.8}
$$

The above two estimates will play an interesting role later. We now introduce the following definition.

Definition 2.2. A pair of $(Y(\cdot),Z(\cdot\,,\cdot))\in\mathcal{H}^p[0,T]$ is called an *adapted M-solution* of MF-BSVIE (1.10) if (1.10) is satisfied in the Itô sense and the following holds:

$$
Y(t)=\mathbb{E}Y(t)+\int_0^t Z(t,s)dW(s),\qquad t\in[0,T].
\tag{2.9}
$$

It is clear that (2.9) implies

$$
Y(t)=\mathbb{E}[Y(t)\mid\mathcal{F}_S]+\int_S^t Z(t,s)dW(s),\qquad 0\le S\le t\le T.
\tag{2.10}
$$

This suggests us define $\mathcal{M}^p[0,T]$ as the set of all elements $(y(\cdot),z(\cdot\,,\cdot))\in\mathcal{H}^p[0,T]$ satisfying:

$$
y(t)=\mathbb{E}\Big[y(t)\mid\mathcal{F}_S\Big]+\int_S^t z(t,s)dW(s),\qquad t\in[S,T],\quad S\in[0,T).
\tag{2.11}
$$

Obviously $\mathcal{M}^p[0,T]$ is a closed subspace of $\mathcal{H}^p[0,T]$. Note that for any $(y(\cdot),z(\cdot\,,\cdot))\in\mathcal{M}^2[0,T]$,

$$
\mathbb{E}|y(t)|^2=\Big(\mathbb{E}\Big[y(t)\mid\mathcal{F}_S\Big]\Big)^2+\mathbb{E}\int_S^t|z(t,s)|^2ds\ge\mathbb{E}\int_S^t|z(t,s)|^2ds.
\tag{2.12}
$$

Relation (2.12) can be generalized a little bit more. To see this, let us present the following lemma.

Lemma 2.3. *Let $0\le S<t\le T$, $\eta\in L^p_{\mathcal{F}_S}(\Omega;\mathbb{R}^n)$ and $\zeta(\cdot)\in L^p_{\mathbb{F}}(\Omega;L^2(S,t;\mathbb{R}^n))$. Then*

$$
\mathbb{E}\Big[|\eta|^p+\Big(\int_S^t|\zeta(s)|^2ds\Big)^{\frac{p}{2}}\Big]\le K\mathbb{E}\Big|\eta+\int_S^t\zeta(s)dW(s)\Big|^p.
\tag{2.13}
$$

Hereafter, $K>0$ stands for a generic constant which can be different from line to line.

Proof. For fixed $(S,t)\in\Delta$ (which means $0\le S\le t\le T$) with $S<t$, let

$$
\xi=\eta+\int_S^t\zeta(s)dW(s),
$$

which is $\mathcal{F}_t$-measurable. Let $(Y(\cdot),Z(\cdot))$ be the adapted solution to the following BSDE:

$$Y(r)=\xi-\int_r^t Z(s)dW(s),\qquad r\in[S,t].$$

Then it is standard that

$$\mathbb{E}\Big[\sup_{r\in[S,t]}|Y(r)|^p+\Big(\int_S^t|Z(s)|^2ds\Big)^{\frac{p}{2}}\Big]\le K\mathbb{E}|\xi|^p. \tag{2.14}$$

Now,

$$Y(S)+\int_S^t Z(s)dW(s)=\xi=\eta+\int_S^t\zeta(s)dW(s).$$

By taking conditional expectation $\mathbb{E}[\,\cdot\,|\,\mathcal{F}_S]$, we see that

$$Y(S)=\eta.$$

Consequently,

$$\int_S^t\Big(Z(s)-\zeta(s)\Big)dW(s)=0,$$

which leads to

$$Z(s)=\zeta(s),\qquad s\in[S,t],\ \text{a.s.}$$

Then (2.13) follows from (2.14). $\square$

We have the following interesting corollary for elements in $\mathcal{M}^p[0,T]$ (comparing with (2.12)).

Corollary 2.4. *For any $(y(\cdot),z(\cdot\,,\cdot))\in\mathcal{M}^p[0,T]$, the following holds:*

$$\mathbb{E}\Big(\int_S^t|z(t,s)|^2ds\Big)^{\frac{p}{2}}\le K\mathbb{E}|y(t)|^p,\qquad\forall S\in[0,t]. \tag{2.15}$$

Proof. Applying (2.13) to (2.11), we have

$$\mathbb{E}\Big(\int_S^t|z(t,s)|^2ds\Big)^{\frac{p}{2}}\le\mathbb{E}\Big[|\mathbb{E}[y(t)\mid\mathcal{F}_S]|^p+\Big(\int_S^t|z(t,s)|^2ds\Big)^{\frac{p}{2}}\Big]\le K\mathbb{E}|y(t)|^p.$$

This proves the corollary. $\square$

From the above, we see that for any $(y(\cdot),z(\cdot,\cdot))\in\mathcal{M}^p[0,T]$, and any $\beta>0$,

$$\begin{aligned}K\mathbb{E}\int_0^T e^{\beta t}|y(t)|^pdt&\ge\mathbb{E}\int_0^T e^{\beta t}\Big[|\mathbb{E}y(t)|^p+\Big(\int_0^t|z(t,s)|^2ds\Big)^{\frac{p}{2}}\Big]dt\\&\ge\mathbb{E}\int_0^T e^{\beta t}\Big(\int_0^t|z(t,s)|^2ds\Big)^{\frac{p}{2}}dt.\end{aligned} \tag{2.16}$$

Hence,

$$\begin{aligned}&\|(y(\cdot),z(\cdot\,,\cdot))\|^p_{\mathcal{H}^p[0,T]}\equiv\mathbb{E}\Big[\int_0^T|y(t)|^pdt+\int_0^T\Big(\int_0^T|z(t,s)|^2ds\Big)^{\frac{p}{2}}dt\Big]\\&\le K\mathbb{E}\Big[\int_0^T|y(t)|^pdt+\int_0^T\Big(\int_0^t|z(t,s)|^2ds\Big)^{\frac{p}{2}}dt+\int_0^T\Big(\int_t^T|z(t,s)|^2ds\Big)^{\frac{p}{2}}dt\Big]\\&\le K\mathbb{E}\Big[\int_0^T e^{\beta t}|y(t)|^pdt+\int_0^T e^{\beta t}\Big(\int_0^t|z(t,s)|^2ds\Big)^{\frac{p}{2}}dt+\int_0^T e^{\beta t}\Big(\int_t^T|z(t,s)|^2ds\Big)^{\frac{p}{2}}dt\Big]\\&\le K\mathbb{E}\Big[\int_0^T e^{\beta t}|y(t)|^pdt+\int_0^T e^{\beta t}\Big(\int_t^T|z(t,s)|^2ds\Big)^{\frac{p}{2}}dt\Big]\le K\|(y(\cdot),z(\cdot\,,\cdot))\|^p_{\mathcal{H}^p[0,T]}.\end{aligned}$$

This means that we can use the following as an equivalent norm in $\mathcal{M}^p[0,T]$:

$$\|(y(\cdot),z(\cdot\,,\cdot))\|_{\mathcal{M}^p[0,T]}\equiv\Big\{\mathbb{E}\int_0^T e^{\beta t}|y(t)|^p dt+\mathbb{E}\int_0^T e^{\beta t}\Big(\int_t^T|z(t,s)|^2ds\Big)^{\frac{p}{2}}dt\Big\}^{\frac{1}{p}}.$$

Sometimes we use $\mathcal{M}^p_\beta[0,T]$ for $\mathcal{M}^p[0,T]$ to emphasize the involved parameter β.

To conclude this subsection, we state the following corollary of Lemma 2.3 relevant to BSVIEs, whose proof is straightforward.

Corollary 2.5. *Suppose $(\eta(\cdot),\zeta(\cdot,\cdot))$ is an adapted M-solution to the following BSVIE:*

$$\eta(t)=\xi(t)+\int_t^T g(t,s)ds-\int_t^T\zeta(t,s)dW(s),\quad t\in[0,T],\tag{2.17}$$

for $\xi(\cdot)\in L^p_{\mathcal{F}_T}(0,T;\mathbb{R}^n)$ and $g(\cdot\,,\cdot)\in L^p(0,T;L^1_{\mathbb{F}}(0,T;\mathbb{R}^n))$. Then

$$\mathbb{E}\Big[|\eta(t)|^p+\Big(\int_t^T|\zeta(t,s)|^2ds\Big)^{\frac{p}{2}}\Big]\le K\mathbb{E}\Big[|\xi(t)|^p+\Big(\int_t^T|g(t,s)|ds\Big)^p\Big],\quad\forall t\in[0,T].\tag{2.18}$$

2.2 Mean-field forward stochastic Volterra integral equations.

In this subsection, we study the following MF-FSVIE:

$$\begin{aligned}X(t)=\varphi(t)&+\int_0^t b(t,s,X(s),\Gamma^b(t,s,X(s)))ds\\&+\int_0^t\sigma(t,s,X(s),\Gamma^\sigma(t,s,X(s)))dW(s),\qquad t\in[0,T],\end{aligned}\tag{2.19}$$

where

$$\left\{\begin{aligned}&\Gamma^b(t,s,X)=\mathbb{E}'\Big[\theta^b(t,s,\xi,X,)\Big]_{\xi=X}\equiv\int_\Omega\theta^b(t,s,\omega,\omega',X(\omega),X(\omega'))\mathbb{P}(d\omega'),\\&\Gamma^\sigma(t,s,X)=\mathbb{E}'\Big[\theta^\sigma(t,s,\xi,X)\Big]_{\xi=X}\equiv\int_\Omega\theta^\sigma(t,s,\omega,\omega',X(\omega),X(\omega'))\mathbb{P}(d\omega').\end{aligned}\right.\tag{2.20}$$

We see that MF-FSVIE (2.19) is slightly more general than MF-FSVIE (1.6) because of the above definition (2.20) of the operators Γ^b and Γ^σ.

An $\mathbb{F}$-adapted process $X(\cdot)$ is called a solution to (2.19) if (2.19) is satisfied in the usual Itô sense. To guarantee the well-posedness of (2.19), let us make the following hypotheses.

(H1) The maps $b:\Delta^*\times\Omega\times\mathbb{R}^n\times\mathbb{R}^{m_1}\to\mathbb{R}^n$ and $\sigma:\Delta^*\times\Omega\times\mathbb{R}^n\times\mathbb{R}^{m_2}\to\mathbb{R}^n$ are measurable, and for all $(t,x,\gamma,\gamma')\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}$, the map

$$(s,\omega)\mapsto(b(t,s,\omega,x,\gamma),\sigma(t,s,\omega,x,\gamma'))$$

is $\mathbb{F}$-progressively measurable on $[0,t]$. Moreover, there exists some constant $L>0$ such that

$$\begin{aligned}&|b(t,s,\omega,x_1,\gamma_1)-b(t,s,\omega,x_2,\gamma_2)|+|\sigma(t,s,\omega,x_1,\gamma_1')-\sigma(t,s,\omega,x_2,\gamma_2')|\\&\quad\le L(|x_1-x_2|+|\gamma_1-\gamma_2|+|\gamma_1'-\gamma_2'|),\\&\qquad\qquad(t,s,\omega)\in\Delta^*\times\Omega,\ (x_i,\gamma_i,\gamma_i')\in\mathbb{R}^n\times\mathbb{R}^{m_1}\times\mathbb{R}^{m_2},\ i=1,2.\end{aligned}\tag{2.21}$$

Moreover,

$$\begin{aligned}&|b(t,s,\omega,x,\gamma)|+|\sigma(t,s,\omega,x,\gamma')|\le L(1+|x|+|\gamma|+|\gamma'|),\\&\qquad\qquad\qquad(t,s,\omega,x,\gamma,\gamma')\in\Delta^*\times\Omega\times\mathbb{R}^n\times\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}.\end{aligned}\tag{2.22}$$

(H2) The maps $\theta^b:\Delta^*\times\Omega^2\times\mathbb{R}^{2n}\to\mathbb{R}^{m_1}$ and $\theta^\sigma:\Delta^*\times\Omega^2\times\mathbb{R}^{2n}\to\mathbb{R}^{m_2}$ are measurable, and for all $(t,x,x')\in[0,T]\times\mathbb{R}^{2n}$, the map

$$(s,\omega,\omega')\mapsto(\theta^b(t,s,\omega,\omega',x,x'),\theta^\sigma(t,s,\omega,\omega',x,x'))$$

is $\mathbb{F}^2$-progressively measurable on $[0,t]$. Moreover, there exists some constant $L>0$ such that

$$\begin{aligned}&|\theta^b(t,s,\omega,\omega',x_1,x_1')-\theta^b(t,s,\omega,\omega',x_2,x_2')|+|\theta^\sigma(t,s,\omega,\omega',x_1,x_1')-\theta^\sigma(t,s,\omega,\omega',x_2,x_2')|\\&\quad\le L(|x_1-x_2|+|x_1'-x_2'|),\qquad(t,s,\omega,\omega')\in\Delta^*\times\Omega^2,(x_i,x_i')\in\mathbb{R}^{2n},i=1,2,\end{aligned}\tag{2.23}$$

and

$$\begin{aligned}&|\theta^b(t,s,\omega,\omega',x,x')|+|\theta^\sigma(t,s,\omega,\omega',x,x')|\le L(1+|x|+|x'|),\\&\qquad\qquad\qquad(t,s,\omega,\omega')\in\Delta^*\times\Omega^2,x,x'\in\mathbb{R}^n.\end{aligned}\tag{2.24}$$

We will also need the following assumptions.

(H1)$'$ In addition to (H1), the map $t\mapsto(b(t,s,\omega,x,\gamma),\sigma(t,s,\omega,x,\gamma'))$ is continuous on $[s,T]$.

(H2)$'$ In addition to (H2), the map $t\mapsto(\theta^b(t,s,\omega,\omega',x,x'),\theta^\sigma(t,s,\omega,\omega',x,x'))$ is continuous on $[s,T]$.

Now, let us state and prove the following result concerning MF-FSVIE (2.19).

Theorem 2.6. *Let* (H1)–(H2) *hold. Then for any $p\ge2$, and $\varphi(\cdot)\in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$, MF-FSVIE* (2.19) *admits a unique solution $X(\cdot)\in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$, and the following estimate holds:*

$$\mathbb{E}\int_0^T|X(t)|^pdt\le K\Big(1+\mathbb{E}\int_0^T|\varphi(t)|^pdt\Big).\tag{2.25}$$

Further, for $i=1,2$, let $X_i(\cdot)\in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$ be the solutions of (2.19) *corresponding to $\varphi_i(\cdot)\in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$ and $b_i(\cdot),\sigma_i(\cdot),\theta_i^b(\cdot),\theta_i^\sigma(\cdot)$ satisfying* (H1)–(H2). *Let*

$$\left\{\begin{aligned}&\Gamma_i^b(t,s,X)=\mathbb{E}'\Big[\theta_i^b(t,s,\xi,X)\Big]_{\xi=X}\equiv\int_\Omega\theta_i^b(t,s,\omega,\omega',X(\omega),X(\omega'))\mathbb{P}(d\omega'),\\&\Gamma_i^\sigma(t,s,X)=\mathbb{E}'\Big[\theta_i^\sigma(t,s,\xi,X)\Big]_{\xi=X}\equiv\int_\Omega\theta_i^\sigma(t,s,\omega,\omega',X(\omega),X(\omega'))\mathbb{P}(d\omega'),\end{aligned}\right.\qquad i=1,2.$$

Then the following stability estimate holds:

$$\begin{aligned}&\mathbb{E}\int_0^T|X_1(t)-X_2(t)|^p\le K\Big\{\mathbb{E}\int_0^T|\varphi_1(t)-\varphi_2(t)|^pdt\\&\quad+\mathbb{E}\int_0^T\Big(\int_0^t|b_1(t,s,X_1(s),\Gamma_1^b(t,s,X_1(s)))-b_2(t,s,X_1(s),\Gamma_2^b(t,s,X_1(s)))|ds\Big)^pdt\\&\quad+\mathbb{E}\int_0^T\Big(\int_0^t|\sigma_1(t,s,X_1(s),\Gamma_1^b(t,s,X_1(s)))-\sigma_2(t,s,X_1(s),\Gamma_2^b(t,s,X_1(s)))|^2ds\Big)^{\frac p2}dt\Big\}.\end{aligned}\tag{2.26}$$

Moreover, let (H1)$'$–(H2)$'$ *hold. Then for any* $p \geq 2$, *and any* $\varphi(\cdot) \in C^p_{\mathbb{F}}([0,T];\mathbb{R}^n)$, *the unique solution* $X(\cdot) \in C^p_{\mathbb{F}}([0,T];\mathbb{R}^n)$, *and estimate* (2.25) *is replaced by the following:*

$$\sup_{t\in[0,T]} \mathbb{E}|X(t)|^p \leq K\Big\{1 + \sup_{t\in[0,T]} \mathbb{E}|\varphi(t)|^p\Big\}. \tag{2.27}$$

Also, for $i = 1,2$, *let* $X_i(\cdot) \in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$ *be the solutions of* (2.19) *corresponding to* $\varphi_i(\cdot) \in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$ *and* $b_i(\cdot), \sigma_i(\cdot), \theta^b_i(\cdot), \theta^\sigma_i(\cdot)$ *satisfying* (H1)$'$–(H2)$'$. *Then* (2.26) *is replaced by the following:*

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}|X_1(t) - X_2(t)|^p \leq K\Big\{\sup_{t\in[0,T]} \mathbb{E}|\varphi_1(t) - \varphi_2(t)|^p \\
&+ \sup_{t\in[0,T]} \mathbb{E}\Big(\int_0^t |b_1(t,s,X_1(s),\Gamma^b_1(t,s,X_1(s))) - b_2(t,s,X_1(s),\Gamma^b_2(t,s,X_1(s)))|ds\Big)^p \\
&+ \sup_{t\in[0,T]} \mathbb{E}\Big(\int_0^t |\sigma_1(t,s,X_1(s),\Gamma^b_1(t,s,X_1(s))) - \sigma_2(t,s,X_1(s),\Gamma^b_2(t,s,X_1(s)))|^2 ds\Big)^{\frac{p}{2}}\Big\}.
\end{aligned} \tag{2.28}$$

Proof. By (H2), similar to (2.7)–(2.8), making use of (2.24), for any $X(\cdot) \in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$, we have

$$|\Gamma^b(t,s,X(s))| + |\Gamma^\sigma(t,s,X(s))| \leq L\Big(1 + \mathbb{E}|X(s)| + |X(s)|\Big). \tag{2.29}$$

Thus, if $X(\cdot) \in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$ is a solution to (2.19) with $\varphi(\cdot) \in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$, then by (2.22),

$$\begin{aligned}
\mathbb{E}|X(t)|^p &\leq 3^{p-1}\mathbb{E}\Big\{|\varphi(t)|^p + \Big|\int_0^t b(t,s,X(s),\Gamma^b(t,s,X(s)))ds\Big|^p \\
&\qquad + \Big|\int_0^t \sigma(t,s,X(s),\Gamma^\sigma(t,s,X(s)))dW(s)\Big|^p\Big\} \\
&\leq 3^{p-1}\Big\{\mathbb{E}|\varphi(t)|^p + \mathbb{E}\Big(\int_0^t L\Big[1 + |X(s)| + |\Gamma^b(t,s,X(s))|\Big]ds\Big)^p \\
&\qquad + \mathbb{E}\Big(\int_0^t L^2\Big[1 + |X(s)| + |\Gamma^\sigma(t,s,X(s))|\Big]^2 ds\Big)^{\frac{p}{2}}\Big\} \\
&\leq K\Big\{1 + \mathbb{E}|\varphi(t)|^p + \int_0^t |X(s)|^p ds\Big\}.
\end{aligned} \tag{2.30}$$

Consequently,

$$\int_0^t \mathbb{E}|X(r)|^p dr \leq K\Big\{1 + \int_0^t \mathbb{E}|\varphi(r)|^p dr + \int_0^t \Big[\int_0^r |X(s)|^p ds\Big]dr\Big\}, \qquad 0 \leq t \leq T.$$

Using Gronwall's inequality, we obtain (2.25).

Now, let $\delta > 0$ be undetermined. For any $x(\cdot) \in L^p_{\mathbb{F}}(0,\delta;\mathbb{R}^n)$, define

$$\begin{aligned}
\mathcal{G}(x(\cdot))(t) &= \varphi(t) + \int_0^t b(t,s,x(s),\Gamma^b(t,s,x(s)))ds \\
&\qquad + \int_0^t \sigma(t,x,x(s),\Gamma^\sigma(t,s,x(s)))dW(s), \qquad t \in [0,\delta].
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mathbb{E}\int_0^\delta |\mathcal{G}(x(\cdot))(t)|^p dt \le K\mathbb{E}\Big\{\int_0^\delta |\varphi(t)|^p dt + \int_0^\delta \Big(\int_0^t \Big(1+|x(s)|+|\Gamma^b(t,s,x(s))|\Big)ds\Big)^p \\
&\qquad\qquad\qquad\qquad + \int_0^\delta \Big|\int_0^t \sigma(t,s,x(s),\Gamma^\sigma(t,s,x(s)))dW(s)\Big|^p dt\Big\} \\
&\le K\Big\{\mathbb{E}\int_0^\delta |\varphi(t)|^p dt + \mathbb{E}\int_0^\delta |x(t)|^p dt\Big\}.
\end{aligned}$$

Thus, $\mathcal{G} : L^p_{\mathbb{F}}(0,\delta;\mathbb{R}^n) \to L^p_{\mathbb{F}}(0,\delta;\mathbb{R}^n)$. Next, for any $x_1(\cdot), x_2(\cdot) \in L^p_{\mathbb{F}}(0,\delta;\mathbb{R}^n)$, we have (making use of (2.21) and (2.23))

$$\begin{aligned}
&\mathbb{E}\int_0^\delta |\mathcal{G}(x_1(\cdot))(t) - \mathcal{G}(x_2(\cdot))(t)|^p dt \\
&\le 2^{p-1}\Big\{\mathbb{E}\int_0^\delta \Big[\int_0^t L\Big(|x_1(s)-x_2(s)| + |\Gamma^b(t,s,x_1(s)) - \Gamma^b(t,s,x_2(s))|\Big)ds\Big]^p dt \\
&\quad + \int_0^\delta \mathbb{E}\Big[\int_0^t L^2\Big(|x_1(s)-x_2(s)|^2 + |\Gamma^\sigma(t,s,x_1(s)) - \Gamma^\sigma(t,s,x_2(s))|^2\Big)ds\Big]^{\frac{p}{2}} dt\Big\} \\
&\le K_0\delta\mathbb{E}\int_0^\delta |x_1(t)-x_2(t)|^p dt,
\end{aligned}$$

with $K_0 > 0$ being an absolute constant (only depending on L and p). Then letting $\delta = \frac{1}{2K_0}$, we see that $\mathcal{G} : L^p_{\mathbb{F}}(0,\delta;\mathbb{R}^n) \to L^p_{\mathbb{F}}(0,\delta;\mathbb{R}^n)$ is a contraction. Hence, MF-FSVIE (2.19) admits a unique solution $X(\cdot) \in L^p_{\mathbb{F}}(0,\delta;\mathbb{R}^n)$.

Next, for $t \in [\delta, 2\delta]$, we write (1.6) as

$$\begin{aligned}
X(t) = \widehat{\varphi}(t) &+ \int_\delta^t b(t,s,X(s),\Gamma^b(t,s,X(s)))ds \\
&+ \int_\delta^t \sigma(t,s,X(s),\Gamma^\sigma(t,s,X(s))dW(s),
\end{aligned} \tag{2.31}$$

with

$$\widehat{\varphi}(t) = \varphi(t) + \int_0^\delta b(t,s,X(s),\Gamma^b(t,s,X(s)))ds + \int_0^\delta \sigma(t,s,X(s),\Gamma^\sigma(t,s,X(s))dW(s).$$

Then a similar argument as above applies to obtain a unique solution of (2.31) on $[\delta, 2\delta]$. It is important to note that the step-length $\delta > 0$ is uniform. Hence, by induction, we obtain the unique solvability of (2.19) on $[0,T]$.

Now, for $i = 1, 2$, let $X_i(\cdot) \in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$ be the solutions of (2.19) corresponding to $\varphi_i(\cdot) \in$

$L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$ and $b_i(\cdot),\sigma_i(\cdot),\theta_i^b(\cdot),\theta_i^\sigma(\cdot)$ (satisfying (H1)–(H2)). Then

$$
\begin{aligned}
&\mathbb{E}|X_1(t)-X_2(t)|^p\le 3^{p-1}\mathbb{E}\Big\{|\varphi_1(t)-\varphi_2(t)|^p\\
&\quad+\Big(\int_0^t|b_1(t,s,X_1(s),\Gamma_1^b(t,s,X_1(s)))-b_2(t,s,X_2(s),\Gamma_2^b(t,s,X_2(s)))|ds\Big)^p\\
&\quad+\Big|\int_0^t\Big(\sigma_1(t,s,X_1(s),\Gamma_1^b(t,s,X_1(s)))-\sigma_2(t,s,X_2(s),\Gamma_2^b(t,s,X_2(s)))\Big)dW(s)\Big|^p\Big\}\\
&\le K\Big\{\mathbb{E}|\varphi_1(t)-\varphi_2(t)|^p+\mathbb{E}\int_0^t|X_1(s)-X_2(s)|^pds\\
&\quad+\mathbb{E}\Big(\int_0^t|b_1(t,s,X_1(s),\Gamma_1^b(t,s,X_1(s)))-b_2(t,s,X_1(s),\Gamma_2^b(t,s,X_1(s)))|ds\Big)^p\\
&\quad+\mathbb{E}\Big|\int_0^t\Big(\sigma_1(t,s,X_1(s),\Gamma_1^b(t,s,X_1(s)))-\sigma_2(t,s,X_1(s),\Gamma_2^b(t,s,X_1(s)))\Big)dW(s)\Big|^p\Big\}.
\end{aligned}
$$

Then we can obtain estimate (2.26).

The conclusions under (H1)$'$–(H2)$'$ are easy to obtain. $\square$

2.3 Linear MF-FSVIEs and MF-BSVIEs.

Let us now look at linear MF-FSVIEs, by which we mean the following:

$$
\begin{aligned}
X(t)=\varphi(t)&+\int_0^t\Big(A_0(t,s)X(s)+\mathbb{E}'\Big[C_0(t,s)X(s)\Big]\Big)ds\\
&+\int_0^t\Big(A_1(t,s)X(s)+\mathbb{E}'\Big[C_1(t,s)X(s)\Big]\Big)dW(s),\qquad t\in[0,T].
\end{aligned}
\tag{2.32}
$$

For such an equation, we introduce the following hypotheses.

(L1) The maps

$$A_0,A_1:\Delta^*\times\Omega\to\mathbb{R}^{n\times n},\quad C_0,C_1:\Delta^*\times\Omega^2\to\mathbb{R}^{n\times n},$$

are measurable and uniformly bounded. For any $t\in[0,T]$, $s\mapsto(A_0(t,s),A_1(t,s))$ is $\mathbb{F}$-progressively measurable on $[0,t]$, and $s\mapsto(C_0(t,s),C_1(t,s))$ is $\mathbb{F}^2$-progressively measurable on $[0,t]$.

(L1)$'$ In addition to (L1), the maps

$$t\mapsto(A_0(t,s,\omega),A_1(t,s,\omega),C_0(t,s,\omega,\omega'),C_1(t,s,\omega,\omega'))$$

is continuous on $[s,T]$.

Clearly, by defining

$$
\left\{
\begin{aligned}
&b(t,s,\omega,x,\gamma)=A_0(t,s,\omega)x+\gamma,\quad \theta^b(t,s,\omega,\omega',x,x')=C_0(t,s,\omega,\omega')x',\\
&\sigma(t,s,\omega,x,\gamma')=A_1(t,s,\omega)x+\gamma',\quad \theta^\sigma(t,s,\omega,\omega',x,x')=C_1(t,s,\omega,\omega')x',
\end{aligned}
\right.
$$

we see that (2.32) is a special case of (2.19). Moreover, (L1) implies (H1)–(H2), and (L1)$'$ implies (H1)$'$–(H2)$'$. Hence, we have the following corollary of Theorem 2.6.

Corollary 2.7. *Let* (L1) *hold, and* $p \geq 2$. *Then for any* $\varphi(\cdot) \in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$, (2.32) *admits a unique solution* $X(\cdot) \in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$, *and estimate* (2.25) *holds. Further, let* $p > 2$. *If for* $i = 1,2$, $A^i_0(\cdot), A^i_1(\cdot), C^i_0(\cdot)$, $C^i_1(\cdot)$ *satisfy* (L1), $\varphi_i(\cdot) \in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$, *and* $X_i(\cdot) \in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$ *are the corresponding solutions to* (2.32), *then for any* $r \in (2,p)$,

$$
\begin{aligned}
&\mathbb{E}\int_0^T |X_1(t)-X_2(t)|^r dt \leq K\mathbb{E}\int_0^T |\varphi_1(t)-\varphi_2(t)|^r dt + K\Big(1+\mathbb{E}\int_0^T |\varphi_1(t)|^p dt\Big)^{\frac{r}{p}}\\
&\cdot\int_0^T\Big\{\Big[\mathbb{E}\Big(\int_0^t |A^1_0(t,s)-A^2_0(t,s)|^{\frac{r}{r-1}}ds\Big)^{\frac{(r-1)p}{p-r}}\Big]^{\frac{p-r}{p}}+\Big[\mathbb{E}^2\Big(\int_0^t |C^1_0(t,s)-C^2_0(t,s)|^{\frac{r}{r-1}}ds\Big)^{\frac{(r-1)p}{p-r}}\Big]^{\frac{p-r}{p}}\\
&+\Big[\mathbb{E}\Big(\int_0^t |A^1_1(t,s)-A^2_1(t,s)|^{\frac{2r}{r-2}}ds\Big)^{\frac{(r-2)p}{2(p-r)}}\Big]^{\frac{p-r}{p}}+\Big[\mathbb{E}^2\Big(\int_0^t |C^1_1(t,s)-C^2_1(t,s)|^{\frac{2r}{r-2}}ds\Big)^{\frac{(r-2)p}{2(p-r)}}\Big]^{\frac{p-r}{p}}\Big\}dt.
\end{aligned}
\tag{2.33}
$$

Moreover, let (L1)$'$ *hold. Then for any* $\varphi(\cdot) \in C^p_{\mathbb{F}}([0,T];\mathbb{R}^n)$, (2.32) *admits a unique solution* $X(\cdot) \in C^p_{\mathbb{F}}([0,T];\mathbb{R}^n)$, *and estimate* (2.27) *holds. Now for* $i = 1,2$, *let* $A^i_0(\cdot), A^i_1(\cdot), C^i_0(\cdot)$, $C^i_1(\cdot)$ *satisfy* (L1)$'$, $\varphi_i(\cdot) \in C^p_{\mathbb{F}}([0,T];\mathbb{R}^n)$, *and* $X_i(\cdot) \in C^p_{\mathbb{F}}([0,T];\mathbb{R}^n)$ *be the corresponding solutions to* (2.32), *then for any* $2 < r < p$,

$$
\begin{aligned}
&\sup_{t\in[0,T[}\mathbb{E}|X_1(t)-X_2(t)|^r \leq K\sup_{t\in[0,T]}\mathbb{E}|\varphi_1(t)-\varphi_2(t)|^r\\
&\quad+K\Big(1+\sup_{t\in[0,T]}\mathbb{E}|\varphi_1(t)|^p\Big)^{\frac{r}{p}}\Big\{\sup_{t\in[0,T]}\Big[\mathbb{E}\Big(\int_0^t |A^1_0(t,s)-A^2_0(t,s)|^{\frac{r}{r-1}}ds\Big)^{\frac{(r-1)p}{p-r}}\Big]^{\frac{p-r}{p}}\\
&\quad+\sup_{t\in[0,T]}\Big[\mathbb{E}^2\Big(\int_0^t |C^1_0(t,s)-C^2_0(t,s)|^{\frac{r}{r-1}}ds\Big)^{\frac{(r-1)p}{p-r}}\Big]^{\frac{p-r}{p}}\\
&\quad+\sup_{t\in[0,T]}\Big[\mathbb{E}\Big(\int_0^t |A^1_1(t,s)-A^2_1(t,s)|^{\frac{2r}{r-2}}ds\Big)^{\frac{(r-2)p}{2(p-r)}}\Big]^{\frac{p-r}{p}}\\
&\quad+\sup_{t\in[0,T]}\Big[\mathbb{E}^2\Big(\int_0^t |C^1_1(t,s)-C^2_1(t,s)|^{\frac{2r}{r-2}}ds\Big)^{\frac{(r-2)p}{2(p-r)}}\Big]^{\frac{p-r}{p}}\Big\}.
\end{aligned}
\tag{2.34}
$$

Proof. We need only to prove the stability estimate. Let $X_i(\cdot) \in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$ be the solutions to the linear MF-FSVIEs corresponding to the coefficients $(A^i_0(\cdot), C^i_0(\cdot), A^i_1(\cdot), C^i_1(\cdot))$ satisfying (L1) and free term $\varphi_i(\cdot) \in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$. Then we have

$$
\mathbb{E}\int_0^T |X_i(s)|^p ds \leq K\Big(1+\mathbb{E}\int_0^T |\varphi_i(s)|^p\Big).
$$

Now, for any $2<r<p$,

$$
\begin{aligned}
&\mathbb{E}\int_0^T|X_1(t)-X_2(t)|^r\le K\Big\{\mathbb{E}\int_0^T|\varphi_1(t)-\varphi_2(t)|^r dt\\
&+\mathbb{E}\int_0^T\Big[\int_0^t\Big(|A_0^1(t,s)-A_0^2(t,s)||X_1(s)|+\mathbb{E}'\big[\,|C_0^1(t,s)-C_0^2(t,s)||X_1(s)|\,\big]\Big)ds\Big]^r dt\\
&+\mathbb{E}\int_0^T\Big[\int_0^t\Big(|A_1^1(t,s)-A_1^2(t,s)|^2|X_1(s)|^2+\mathbb{E}'\big[\,|C_1^1(t,s)-C_1^2(t,s)|^2|X_1(s)|^2\,\big]\Big)ds\Big]^{\frac{r}{2}}dt\Big\}\\
&\le K\Big\{\mathbb{E}\int_0^T|\varphi^1(t)-\varphi^2(t)|^r dt+\mathbb{E}\int_0^T\Big(\int_0^t|A_0^1(t,s)-A_0^2(t,s)||X_1(s)|ds\Big)^r dt\\
&\quad+\mathbb{E}^2\int_0^T\Big(\int_0^t|C_0^1(t,s)-C_0^2(t,s)||X_1(s)|ds\Big)^r dt+\mathbb{E}\int_0^T\Big(\int_0^t|A_1^1(t,s)-A_1^2(t,s)|^2|X_1(s)|^2ds\Big)^{\frac{r}{2}}dt\\
&\quad+\mathbb{E}^2\int_0^T\Big(\int_0^t|C_1^1(t,s)-C_1^2(t,s)|^2|X_1(s)|^2ds\Big)^{\frac{r}{2}}dt\Big\}\\
&\le K\Big\{\mathbb{E}\int_0^T|\varphi^1(t)-\varphi^2(t)|^r dt+\mathbb{E}\int_0^T\Big(\int_0^t|A_0^1(t,s)-A_0^2(t,s)|^{\frac{r}{r-1}}ds\Big)^{r-1}\Big(\int_0^t|X_1(s)|^r ds\Big)dt\\
&\quad+\mathbb{E}^2\int_0^T\Big(\int_0^t|C_0^1(t,s)-C_0^2(t,s)|^{\frac{r}{r-1}}ds\Big)^{r-1}\Big(\int_0^t|X_1(s)|^r ds\Big)dt\\
&\quad+\mathbb{E}\int_0^T\Big(\int_0^t|A_1^1(t,s)-A_1^2(t,s)|^{\frac{2r}{r-2}}ds\Big)^{\frac{r-2}{2}}\Big(\int_0^t|X_1(s)|^r ds\Big)dt\\
&\quad+\mathbb{E}^2\int_0^T\Big(\int_0^t|C_1^1(t,s)-C_1^2(t,s)|^{\frac{2r}{r-2}}ds\Big)^{\frac{r-2}{2}}\Big(\int_0^t|X_1(s)|^r ds\Big)dt\Big\}\\
&\le K\mathbb{E}\int_0^T|\varphi^1(t)-\varphi^2(t)|^r dt+K\mathbb{E}\int_0^T\Big[\mathbb{E}\Big(\int_0^t|X_1(s)|^r ds\Big)^{\frac{p}{r}}\Big]^{\frac{r}{p}}\\
&\quad\cdot\Big\{\Big[\mathbb{E}\Big(\int_0^t|A_0^1(t,s)-A_0^2(t,s)|^{\frac{r}{r-1}}ds\Big)^{\frac{(r-1)p}{p-r}}\Big]^{\frac{p-r}{p}}+\Big[\mathbb{E}^2\Big(\int_0^t|C_0^1(t,s)-C_0^2(t,s)|^{\frac{r}{r-1}}ds\Big)^{\frac{(r-1)p}{p-r}}\Big]^{\frac{p-r}{p}}\\
&\quad+\Big[\mathbb{E}\Big(\int_0^t|A_1^1(t,s)-A_1^2(t,s)|^{\frac{2r}{r-2}}ds\Big)^{\frac{(r-2)p}{2(p-r)}}\Big]^{\frac{p-r}{p}}+\Big[\mathbb{E}^2\Big(\int_0^t|C_1^1(t,s)-C_1^2(t,s)|^{\frac{2r}{r-2}}ds\Big)^{\frac{(r-2)p}{2(p-r)}}\Big]^{\frac{p-r}{p}}\Big\}dt.
\end{aligned}
$$

Then (2.33) follows. The case that (L1)$'$ holds case be proved similarly. $\square$

We point out that linear MF-FSVIE (2.32) is general enough in some sense. To see this, let us formally look at the variational equation of (2.19). More precisely, let $X^\delta(\cdot)$ be the unique solution of (2.19) with $\varphi(\cdot)$ replaced by $\varphi(\cdot)+\delta\bar\varphi(\cdot)$. We formally let

$$
\bar X(t)=\lim_{\delta\to0}\frac{X^\delta(t)-X(t)}{\delta}.
$$

Then $\bar X(\cdot)$ should satisfy the following linear MF-FSVIE:

$$
\begin{aligned}
\bar X(t)=\bar\varphi(t)&+\int_0^t\Big(b_x(t,s)\bar X(s)+b_\gamma(t,s)\mathbb{E}'\Big[\theta_x^b(t,s)\bar X(s,\omega)+\theta_{x'}^b(t,s)\bar X(s,\omega')\Big]\Big)ds\\
&+\int_0^t\Big(\sigma_x(t,s)\bar X(s)+\sigma_\gamma(t,s)\mathbb{E}'\Big[\theta_x^\sigma(t,s)\bar X(s,\omega)+\theta_{x'}^\sigma(t,s)\bar X(s,\omega')\Big]\Big)dW(s),
\end{aligned}\tag{2.35}
$$

where (with a little misuse of γ)

$$
\left\{\begin{aligned}
&b_x(t,s)=b_x(t,s,\omega,\Gamma^b(t,s,X(s,\omega))),\quad \theta_x^b(t,s)=\theta_x^b(t,s,\omega,\omega',X(s,\omega),X(s,\omega')),\\
&b_\gamma(t,s)=b_\gamma(t,s,\omega,\Gamma^b(t,s,X(s,\omega))),\quad \theta_{x'}^b(t,s)=\theta_{x'}^b(t,s,\omega,\omega',X(s,\omega),X(s,\omega')),\\
&\sigma_x(t,s)=\sigma_x(t,s,\omega,\Gamma^\sigma(t,s,X(s,\omega))),\quad \theta_x^\sigma(t,s)=\theta_x^\sigma(t,s,\omega,\omega',X(s,\omega),X(s,\omega')),\\
&\sigma_\gamma(t,s)=\sigma_\gamma(t,s,\omega,\Gamma^\sigma(t,s,X(s,\omega))),\quad \theta_{x'}^\sigma(t,s)=\theta_{x'}^\sigma(t,s,\omega,\omega',X(s,\omega),X(s,\omega')).
\end{aligned}\right.\tag{2.36}
$$

It is interesting to note that (2.35) can be written as follows:

$$\begin{aligned}\bar X(t)=\bar\varphi(t)&+\int_0^t\Big\{\Big(b_x(t,s)+b_\gamma(t,s)\mathbb{E}'[\theta_x^b(t,s)]\Big)\bar X(s)+\mathbb{E}'\Big[b_\gamma(t,s)\theta_{x'}^b(t,s)\bar X(s)\Big]\Big\}ds\\&+\int_0^t\Big\{\Big(\sigma_x(t,s)+\sigma_\gamma(t,s)\mathbb{E}'[\theta_x^\sigma(t,s)]\Big)\bar X(s,\omega)+\mathbb{E}'\Big[\sigma_\gamma(t,s)\theta_{x'}^\sigma(t,s)\bar X(s,\omega')\Big]\Big\}dW(s),\end{aligned}\tag{2.37}$$

which is a special case of (2.32).

Mimicking the above, we see that general linear MF-BSVIE should take the following form:

$$\begin{aligned}Y(t)=\psi(t)+\int_t^T&\Big(\bar A_0(t,s)Y(s)+\bar B_0(t,s)Z(t,s)+\bar C_0(t,s)Z(s,t)\\&+\mathbb{E}'\Big[\bar A_1(t,s)Y(s)+\bar B_1(t,s)Z(t,s)+\bar C_1(t,s)Z(s,t)\Big]\Big)ds-\int_t^T Z(t,s)dW(s).\end{aligned}\tag{2.38}$$

For the coefficients, we should adopt the following hypothesis.

(L2) The maps

$$\bar A_0,\bar B_0,\bar C_0:\Delta\times\Omega\to\mathbb{R}^{n\times n},\quad \bar A_1,\bar B_1,\bar C_1:\Delta\times\Omega^2\to\mathbb{R}^{n\times n}$$

are measurable and uniformly bounded. Moreover, for any $t\in[0,T]$, $s\mapsto(\bar A_0(t,s),\bar B_0(t,s),\bar C_0(t,s))$ is $\mathbb{F}$-progressively measurable on $[t,T]$, and $s\mapsto(\bar A_1(t,s),\bar B_1(t,s),\bar C_1(t,s))$ is $\mathbb{F}^2$-progressively measurable on $[t,T]$.

We expect that under (L2), for reasonable $\psi(\cdot)$, the above (2.38) will have a unique adapted M-solution. Such a result will be a consequence of the main result of the next section.

3 Well-posedness of MF-BSVIEs.

In this section, we are going to establish the well-posedness of our MF-BSVIEs. To begin with, let us introduce the following hypothesis.

(H3)$_q$ The map $\theta:\Delta\times\Omega^2\times\mathbb{R}^{6n}\to\mathbb{R}^m$ satisfies (H0)$_q$. The map $g:\Delta\times\Omega\times\mathbb{R}^{3n}\times\mathbb{R}^m\to\mathbb{R}^n$ is measurable and for all $(t,y,z,\widehat z,\gamma)\in[0,T]\times\mathbb{R}^{3n}\times\mathbb{R}^m$, the map $(s,\omega)\mapsto g(t,s,\omega,y,z,\widehat z,\gamma)$ is $\mathbb{F}$-progressively measurable. Moreover, there exist constants $L>0$ and $q\ge2$ such that

$$\begin{aligned}&|g(t,s,\omega,y_1,z_1,\widehat z_1,\gamma_1)-g(t,s,\omega,y_2,z_2,\widehat z_2,\gamma_2)|\\&\le L\Big(|y_1-y_2|+|z_1-z_2|+|\widehat z_1-\widehat z_2|+|\gamma_1-\gamma_2|\Big),\\&\qquad\qquad\forall(t,s,\omega)\in\Delta\times\Omega,(y_i,z_i,\widehat z_i,\gamma_i)\in\mathbb{R}^{3n}\times\mathbb{R}^m,i=1,2,\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}&|g(t,s,\omega,y,z,\widehat z,\gamma)|\le L\Big(1+|y|+|z|+|\widehat z|^{\frac{2}{q}}+|\gamma|\Big),\\&\qquad\qquad\forall(t,s,\omega)\in\Delta\times\Omega,(y,z,\widehat z,\gamma)\in\mathbb{R}^{3n}\times\mathbb{R}^m.\end{aligned}\tag{3.2}$$

Similar to (H0)$_\infty$ in the previous section, (H3)$_\infty$ is understood that (2.5) holds and (3.2) is replaced by the following:

$$\begin{aligned}&|g(t,s,\omega,y,z,\widehat z,\gamma)|\le L\Big(1+|y|+|z|+|\gamma|\Big),\\&\qquad\qquad\forall(t,s,\omega)\in\Delta\times\Omega,(y,z,\widehat z,\gamma)\in\mathbb{R}^{3n}\times\mathbb{R}^m.\end{aligned}\tag{3.3}$$

3.1 A special MF-BSVIE.

In this subsection, we firstly consider the following special MF-BSVIE:

$$Y(t)=\psi(t)+\int_t^T\widetilde g(t,s,Z(t,s),\widetilde\Gamma(t,s,Z(t,s)))ds-\int_t^T Z(t,s)dW(s),\quad t\in[0,T],\tag{3.4}$$

where

$$\left\{\begin{array}{l}\widetilde g(t,s,Z,\gamma)=g(t,s,y(s),Z,z(s,t),\gamma),\\ \widetilde\Gamma(t,s,Z)=\Gamma(t,s,y(s),Z,z(s,t))\equiv\mathbb{E}'\Big[\theta(t,s,y(s),Z,z(s,t),y',z',\widehat z')\Big]_{(y',z',\widehat z')=(y(s),Z,z(s,t))},\end{array}\right.$$

for some given $(y(\cdot),z(\cdot\,,\cdot))\in\mathcal{M}^p[0,T]$. Therefore,

$$\widetilde g(t,s,Z(t,s),\widetilde\Gamma(t,s,Z(t,s)))=g(t,s,y(s),Z(t,s),z(s,t),\Gamma(t,s,y(s),Z(t,s),z(s,t))).$$

Note that we may take much more general $\widetilde g(\cdot)$ and $\widetilde\Gamma(\cdot)$. But the above is sufficient for our purpose, and by restricting such a case, we avoid stating a lengthy assumption similar to $(\mathrm{H3})_q$. We now state and prove the following result concerning MF-BSVIE (3.4).

Proposition 3.1. *Let $(\mathrm{H3})_q$ hold. Then for any $p>1$ and $\psi(\cdot)\in L^p_{\mathcal{F}_T}(0,T;\mathbb{R}^n)$, MF-BSVIE (3.4) admits a unique M-solution $(Y(\cdot),Z(\cdot\,,\cdot))\in\mathcal{M}^p[0,T]$. Moreover, the following estimate holds:*

$$\mathbb{E}\Big[|Y(t)|^p+\Big(\int_t^T|Z(t,s)|^2ds\Big)^{\frac p2}\Big]\le K\mathbb{E}\Big[|\psi(t)|^p+\Big(\int_t^T|\widetilde g(t,s,0,\widetilde\Gamma(t,s,0))|ds\Big)^p\Big].\tag{3.5}$$

Further, for $i=1,2$, let $\psi_i(\cdot)\in L^p_{\mathcal{F}_T}(0,T;\mathbb{R}^n)$, $(y_i(\cdot),z_i(\cdot\,,\cdot))\in\mathcal{M}^p[0,T]$, and

$$\left\{\begin{array}{l}\widetilde g_i(t,s,Z(t,s),\widetilde\Gamma_i(t,s,Z(t,s))=g_i(t,s,y_i(s),Z(t,s),z_i(s,t),\Gamma_i(t,s,y_i(s),Z(t,s),z_i(s,t)),\\ \Gamma_i(t,s,Y,Z,\widehat Z)=\mathbb{E}'\Big[\theta_i(t,s,y_i(s),Z,z_i(s,t),y',z',\widehat z')\Big]_{(y',z',\hat z')=(y_i(s),Z,z_i(s,t))}\end{array}\right.$$

with $g_i(\cdot)$ and $\theta_i(\cdot)$ satisfying $(\mathrm{H3})_q$. Then the corresponding M-solutions $(Y_i(\cdot),Z_i(\cdot))$ satisfy the following stability estimate:

$$\begin{aligned}&\mathbb{E}\Big[|Y_1(t)-Y_2(t)|^p+\Big(\int_t^T|Z_1(t,s)-Z_2(t,s)|^2ds\Big)^{\frac p2}\Big]\le K\mathbb{E}\Big[|\psi_1(t)-\psi_2(t)|^p\\&\qquad+\Big(\int_t^T|\widetilde g_1(t,s,Z_1(t,s),\widetilde\Gamma_1(t,s,Z_1(t,s))-\widetilde g_2(t,s,Z_1(t,s),\widetilde\Gamma_2(t,s,Z_1(t,s))|ds\Big)^p\Big].\end{aligned}\tag{3.6}$$

Proof. Fix $t\in[0,T)$. Consider the following MF-BSDE (parameterized by t):

$$\eta(r)=\psi(t)+\int_r^T\widetilde g(t,s,\zeta(s),\widetilde\Gamma(t,s,\zeta(s)))ds-\int_r^T\zeta(s)dW(s),\quad r\in[t,T].\tag{3.7}$$

If $p\in(1,2]$, it follows from $(\mathrm{H3})_q$ that

$$\begin{aligned}&\mathbb{E}\int_0^T\Big(\int_t^T|\widetilde g(t,s,0,\widetilde\Gamma(t,s,0))|ds\Big)^pdt\\&\le K\mathbb{E}\int_0^T\Big(\int_t^T(|y(s)|+|z(s,t)|+\mathbb{E}|y(s)|+\mathbb{E}|z(s,t)|)ds\Big)^pdt+K\\&\le K\mathbb{E}\int_0^T\int_t^T|y(s)|^pdsdt+K\mathbb{E}\int_0^T\Big(\int_0^s|z(s,t)|^2dt\Big)^{\frac p2}ds+K<\infty,\end{aligned}$$

As to the case of $p=q>2$, similarly we have

$$\begin{aligned}
&\mathbb{E}\int_0^T\Big(\int_t^T|\widetilde g(t,s,0,\widetilde\Gamma(t,s,0))|ds\Big)^q dt\\
&\le K\mathbb{E}\int_0^T\Big(\int_t^T(|y(s)|+|z(s,t)|^{\frac{2}{q}}+\mathbb{E}|y(s)|+\mathbb{E}|z(s,t)|^{\frac{2}{q}})ds\Big)^q dt+K\\
&\le K\mathbb{E}\int_0^T\int_t^T|y(s)|^q dsdt+K\mathbb{E}\int_0^T\int_t^T|z(s,t)|^2dsdt+K<\infty.
\end{aligned}$$

Similar to a standard argument for BSDEs, making use of contraction mapping theorem, we can show that the above MF-BSDE admits a unique adapted solution

$$(\eta(\cdot),\zeta(\cdot))\equiv\Big(\eta(\cdot\,;t,\psi(t)),\zeta(\cdot\,;t,\psi(t))\Big).$$

Moreover, the following estimate holds:

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{r\in[t,T]}|\eta(r;t,\psi(t))|^p+\Big(\int_t^T|\zeta(s;t,\psi(t))|^2ds\Big)^{\frac{p}{2}}\Big]\\
&\le K\mathbb{E}\Big[|\psi(t)|^p+\Big(\int_t^T|\widetilde g(t,s,0,\widetilde\Gamma(t,s,0))|ds\Big)^p\Big].
\end{aligned}\tag{3.8}$$

Further, for $i=1,2$, let $\psi_i(\cdot)\in L^p_{\mathcal{F}_T}(0,T;\mathbb{R}^n)$, $(y_i(\cdot),z_i(\cdot\,,\cdot))\in\mathcal{M}^p[0,T]$, and

$$\begin{cases}\widetilde g_i(t,s,Z(t,s),\widetilde\Gamma_i(t,s,Z(t,s))=g_i(t,s,y_i(s),Z(t,s),z_i(s,t),\Gamma_i(t,s,y_i(s),Z(t,s),z_i(s,t)),\\ \Gamma_i(t,s,Y,Z,\widehat Z)=\mathbb{E}'\Big[\theta_i(t,s,y_i(s),Z,z_i(s,t),y',z',\widehat z')\Big]_{(y',z',\hat z')=(y_i(s),Z,z_i(s,t))}\end{cases}$$

with $g_i(\cdot)$ and $\theta_i(\cdot)$ satisfying (H3)$_q$. Then let $(\eta_i(\cdot),\zeta_i(\cdot))$ be the adapted solutions of the corresponding BSDE. It follows that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{r\in[t,T]}|\eta_1(r)-\eta_2(r)|^p\Big]+\mathbb{E}\Big(\int_t^T|\zeta_1(s)-\zeta_2(s)|^2ds\Big)^{\frac{p}{2}}\\
&\le K\mathbb{E}\Big[|\psi_1(t)-\psi_2(t)|^p+\Big(\int_t^T|\widetilde g_1(t,s,\zeta_1(s),\widetilde\Gamma_1(t,s,\zeta_1(s))-\widetilde g_2(t,s,\zeta_1(s),\widetilde\Gamma_2(t,s,\zeta_1(s))|ds\Big)^p\Big].
\end{aligned}\tag{3.9}$$

Now, we define

$$Y(t)=\eta(t;t,\psi(t)),\quad Z(t,s)=\zeta(s;t,\psi(t)),\qquad (t,s)\in\Delta,$$

and $Z(t,s)$ on Δ^c through the martingale representation:

$$Y(t)=\mathbb{E}Y(t)+\int_0^t Z(t,s)dW(s),\qquad t\in[0,T].$$

Then $(Y(\cdot),Z(\cdot\,,\cdot))\in\mathcal{M}^p[0,T]$ is the unique M-solution to (3.4). Estimates (3.5) and (3.6) follows easily from (3.8) and (3.9), respectively. $\square$

Note that the cases that we are interested in are $p=2,q$. We will use them below.

3.2 The general case.

Now, we consider our MF-BSVIEs. For convenience, let us rewrite (1.10) here:

$$\begin{aligned}Y(t)=\psi(t)+&\int_t^T g(t,s,Y(s),Z(t,s),Z(s,t),\Gamma(t,s,Y(s),Z(t,s),Z(s,t)))ds\\&-\int_t^T Z(t,s)dW(s),\qquad\qquad t\in[0,T],\end{aligned}\tag{3.10}$$

with

$$\begin{aligned}&\Gamma(t,s,Y(s),Z(t,s),Z(s,t))=\mathbb{E}'\Big[\theta(t,s,Y(s),Z(t,s),Z(s,t))\Big]\\&=\int_\Omega\theta(t,s,\omega',\omega,Y(s,\omega'),Z(t,s,\omega'),Z(s,t,\omega'),Y(s,\omega),Z(t,s,\omega),Z(s,t,\omega))\mathbb{P}(d\omega').\end{aligned}\tag{3.11}$$

Our main result of this section is the following.

Theorem 3.2. *Let* $(\mathrm{H3})_q$ *hold with* $2\le q<\infty$. *Then for any* $\psi(\cdot)\in L^q_{\mathcal{F}_T}(0,T;\mathbb{R}^n)$, *MF-BSVIE* (3.10) *admits a unique adapted M-solution* $(Y(\cdot),Z(\cdot\,,\cdot))\in\mathcal{M}^q[0,T]$, *and the following estimate holds:*

$$\|(Y(\cdot),Z(\cdot\,,\cdot))\|_{\mathcal{M}^q[0,T]}\le K\Big(1+\|\psi(\cdot)\|_{L^q_{\mathcal{F}_T}(0,T;\mathbb{R}^n)}\Big).\tag{3.12}$$

Moreover, for $i=1,2$, *let* $g_i(\cdot)$ *and* $\theta_i(\cdot)$ *satisfy* $(\mathrm{H3})_q$, *and* $\psi_i(\cdot)\in L^q_{\mathcal{F}_T}(0,T;\mathbb{R}^n)$. *Let* $(Y_i(\cdot),Z_i(\cdot\,,\cdot))\in\mathcal{M}^q[0,T]$ *be the corresponding adapted M-solutions. Then*

$$\begin{aligned}&\|(Y_1(\cdot),Z_1(\cdot\,,\cdot))-(Y_2(\cdot),Z_2(\cdot\,,\cdot))\|^2_{\mathcal{M}^2[0,T]}\\&\le K\mathbb{E}\Big\{\int_0^T|\psi_1(t)-\psi_2(t)|^2dt+\int_0^T\Big(\int_t^T|(g_1-g_2)(t,s)|ds\Big)^2dt\Big\},\end{aligned}\tag{3.13}$$

where

$$\begin{aligned}(g_1-g_2)(t,s)=g_1(t,s,Y_1(s),Z_1(t,s),Z_1(s,t),\Gamma_1(t,s,Y_1(s),Z_1(t,s),Z_1(s,t)))\\-g_2(t,s,Y_1(s),Z_1(t,s),Z_1(s,t),\Gamma_2(t,s,Y_1(s),Z_1(t,s),Z_1(s,t))),\end{aligned}$$

with

$$\begin{aligned}&\Gamma_i(t,s,Y_i(s),Z_i(t,s),Z_i(s,t))\\&=\mathbb{E}'\Big[\theta_i(t,s,Y_i(s),Z_i(t,s),Z_i(s,t),y,z,\widehat{z})\Big]_{(y,z,\hat z)=(Y_i(s),Z_i(t,s),Z_i(s,t))}\\&\equiv\int_\Omega\theta_i(t,s,\omega',\omega,Y_i(s,\omega'),Z_i(t,s,\omega'),Z_i(s,t,\omega'),Y_i(s,\omega),Z_i(t,s,\omega),Z_i(s,t,\omega))\mathbb{P}(d\omega').\end{aligned}$$

Proof. We split the proof into several steps.

Step 1. Existence and uniqueness of M-solutions of (3.10) in $\mathcal{M}^p[0,T]$ with $p\in(1,2]$.

Let $\psi(\cdot)\in L^p_{\mathcal{F}_T}(0,T;\mathbb{R}^n)$ be given. For any $(y(\cdot),z(\cdot\,,\cdot))\in\mathcal{M}^p[0,T]$, we consider the following MF-BSVIE

$$\begin{aligned}Y(t)=\psi(t)+&\int_t^T g(t,s,y(s),Z(t,s),z(s,t),\Gamma(t,s,y(s),Z(t,s),z(s,t)))ds\\&-\int_t^T Z(t,s)dW(s),\qquad\qquad t\in[0,T].\end{aligned}\tag{3.14}$$

According to Proposition 3.1, there exists a unique adapted M-solution $(Y(\cdot), Z(\cdot\,,\cdot)) \in \mathcal{M}^p[0,T]$. Moreover, the following estimate holds: (making use of (2.7) and (2.8))

$$
\begin{aligned}
&\mathbb{E}\Big\{|Y(t)|^p + \Big(\int_t^T |Z(t,s)|^2 ds\Big)^{\frac{p}{2}}\Big\} \\
&\leq K\mathbb{E}\Big\{|\psi(t)|^p + \Big(\int_t^T |g(t,s,y(s),0,z(s,t),\Gamma(t,s,y(s),0,z(s,t))|ds\Big)^p\Big\} \\
&\leq K\mathbb{E}\Big\{1+|\psi(t)|^p + \Big[\int_t^T \Big(|y(s)|+|z(s,t)|+|\Gamma(t,s,y(s),0,z(s,t))|\Big)ds\Big]^p\Big\} \\
&\leq K\mathbb{E}\Big\{1+|\psi(t)|^p+\Big[\int_t^T\Big(|y(s)|+|z(s,t)|+\mathbb{E}|y(s)|+\mathbb{E}|z(s,t)|+|y(s)|+|z(s,t)|\Big)ds\Big]^p\Big\} \\
&\leq K\mathbb{E}\Big\{1+|\psi(t)|^p + \Big[\int_t^T \Big(|y(s)|+|z(s,t)|\Big)ds\Big]^p\Big\} \\
&\leq K\mathbb{E}\Big\{1+|\psi(t)|^p + \int_t^T |y(s)|^p ds + \int_t^T |z(s,t)|^p ds\Big\}.
\end{aligned}
\tag{3.15}
$$

Consequently, (making use of (2.15) for $(y(\cdot), z(\cdot\,,\cdot)) \in \mathcal{M}^p[0,T]$)

$$
\begin{aligned}
&\|(Y(\cdot),Z(\cdot\,,\cdot))\|^p_{\mathcal{H}^p[0,T]} \equiv \mathbb{E}\Big\{\int_0^T |Y(t)|^p dt + \int_0^T \Big(\int_0^T |Z(t,s)|^2 ds\Big)^{\frac{p}{2}} dt\Big\} \\
&\leq K\mathbb{E}\Big\{1+\int_0^T |\psi(t)|^p dt + \int_0^T\int_t^T |y(s)|^p dsdt + \int_0^T\int_t^T |z(s,t)|^p dsdt\Big\} \\
&\leq K\mathbb{E}\Big\{1+\int_0^T |\psi(t)|^p dt + \int_0^T |y(t)|^p dt + \int_0^T\int_0^t |z(t,s)|^p dsdt\Big\} \\
&\leq K\mathbb{E}\Big\{1+\int_0^T |\psi(t)|^p dt + \int_0^T |y(t)|^p dt + \int_0^T \Big(\int_0^t |z(t,s)|^2 ds\Big)^{\frac{p}{2}} dt\Big\} \\
&\leq K\mathbb{E}\Big\{1+\int_0^T |\psi(t)|^p dt + \int_0^T |y(t)|^p dt\Big\} \\
&\leq K\Big\{1+\|\psi(\cdot)\|^p_{L^p_{\mathcal{F}_T}(0,T;\mathbb{R}^n)} + \|(y(\cdot),z(\cdot\,,\cdot))\|_{\mathcal{M}^p[0,T]}\Big\}.
\end{aligned}
$$

Hence, if we define $\Theta(y(\cdot), z(\cdot\,,\cdot)) = (Y(\cdot), Z(\cdot\,,\cdot))$, then Θ maps from $\mathcal{M}^p[0,T]$ to itself. We now show that the mapping Θ is contractive. To this end, take any $(y_i(\cdot), z_i(\cdot\,,\cdot)) \in \mathcal{M}^p[0,T]$ $(i = 1,2)$, and let

$$(Y_i(\cdot), Z_i(\cdot\,,\cdot)) = \Theta(y_i(\cdot), z_i(\cdot\,,\cdot)).$$

Then by Proposition 3.1, we have (note (2.8))

$$
\begin{aligned}
&\mathbb{E}\Big[|Y_1(t)-Y_2(t)|^p+\Big(\int_t^T|Z_1(t,s)-Z_2(t,s)|^2ds\Big)^{\frac{p}{2}}\Big]\\
&\le K\mathbb{E}\Big(\int_t^T|g(t,s,y_1(s),Z_1(t,s),z_1(s,t),\Gamma(t,s,y_1(s),Z_1(t,s),z_1(s,t)))\\
&\qquad -g(t,s,y_2(s),Z_1(t,s),z_2(s,t),\Gamma(t,s,y_2(s),Z_1(t,s),z_2(s,t)))|ds\Big)^p\\
&\le K\mathbb{E}\Big[\int_t^T\Big(|y_1(s)-y_2(s)|+|z_1(s,t)-z_2(s,t)|\\
&\qquad +|\Gamma(t,s,y_1(s),Z_1(t,s),z_1(s,t))-\Gamma(t,s,y_2(s),Z_1(t,s),z_2(s,t))|\Big)ds\Big]^p\\
&\le K\mathbb{E}\Big[\int_t^T\Big(|y_1(s)-y_2(s)|+|z_1(s,t)-z_2(s,t)|\\
&\qquad +|y_1(s)-y_2(s)|+|z_1(s,t))-z_2(s,t)|+\mathbb{E}|y_1(s)-y_2(s)|+\mathbb{E}|z_1(s,t)-z_2(s,t)|\Big)ds\Big]^p\\
&\le K\mathbb{E}\Big[\int_t^T\Big(|y_1(s)-y_2(s)|+|z_1(s,t)-z_2(s,t)|\Big)ds\Big]^p\\
&\le K\mathbb{E}\Big[\int_t^T|y_1(s)-y_2(s)|^pds+\Big(\int_t^T|z_1(s,t)-z_2(s,t)|ds\Big)^p\Big].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|\Theta(y_1(\cdot),z_1(\cdot\,,\cdot))-\Theta(y_2(\cdot),z_2(\cdot\,,\cdot))\|^p_{\mathcal{M}^p_\beta[0,T]}\\
&\equiv\int_0^Te^{\beta t}\mathbb{E}\Big[|Y_1(t)-Y_2(t)|^p+\Big(\int_t^T|Z_1(t,s)-Z_2(t,s)|^2ds\Big)^{\frac{p}{2}}\Big]dt\\
&\le K\int_0^Te^{\beta t}\mathbb{E}\Big[\int_t^T|y_1(s)-y_2(s)|^pds+\Big(\int_t^T|z_1(s,t)-z_2(s,t)|ds\Big)^p\Big]dt\\
&=K\mathbb{E}\Big[\int_0^T\Big(\int_0^se^{\beta t}dt\Big)|y_1(s)-y_2(s)|^pds+\int_0^Te^{\beta t}\Big(\int_t^Te^{-\frac{\beta}{p}s}e^{\frac{\beta}{p}s}|z_1(s,t)-z_2(s,t)|ds\Big)^pdt\Big]\\
&\le K\mathbb{E}\Big[\frac{1}{\beta}\int_0^Te^{\beta t}|y_1(t)-y_2(t)|^pdt+\int_0^Te^{\beta t}\Big(\int_t^Te^{\frac{-q\beta s}{p}}ds\Big)^{\frac{p}{q}}\Big(\int_t^Te^{\beta s}|z_1(s,t)-z_2(s,t)|^pds\Big)dt\Big]\qquad(3.16)\\
&\le\frac{K}{\beta}\mathbb{E}\int_0^Te^{\beta t}|y_1(t)-y_2(t)|^pdt+K\Big(\frac{p}{\beta q}\Big)^{\frac{p}{q}}\mathbb{E}\int_0^T\int_0^se^{\beta s}|z_1(s,t)-z_2(s,t)|^pdtds\\
&\le\frac{K}{\beta}\mathbb{E}\int_0^Te^{\beta t}|y_1(t)-y_2(t)|^pdt+K\Big(\frac{1}{\beta}\Big)^{\frac{p}{q}}\mathbb{E}\int_0^Te^{\beta s}\Big(\int_0^s|z_1(s,t)-z_2(s,t)|^2dt\Big)^{\frac{p}{2}}ds\\
&\le\Big(\frac{K}{\beta}+K\Big(\frac{1}{\beta}\Big)^{\frac{p}{q}}\Big)\mathbb{E}\int_0^Te^{\beta t}|y_1(t)-y_2(t)|^pdt\\
&\le\Big(\frac{K}{\beta}+K\Big(\frac{1}{\beta}\Big)^{\frac{p}{q}}\Big)\|(y_1(\cdot),z_1(\cdot\,,\cdot))-(y_2(\cdot),z_2(\cdot\,,\cdot))\|^p_{\mathcal{M}^p_\beta[0,T]}.
\end{aligned}
$$

Since the constant $K>0$ appears in the right hand side of the above is independent of β, by choosing $\beta>0$ large, we obtain that Θ is a contraction. Hence, there exists a unique fixed point $(Y(\cdot),Z(\cdot\,,\cdot))\in\mathcal{M}^p[0,T]$, which is the unique adapted M-solution of (1.10).

Step 2. The adapted M-solution $(Y(\cdot),Z(\cdot\,,\cdot))\in\mathcal{M}^q[0,T]$ if $\psi(\cdot)\in L^q_{\mathcal{F}_T}(0,T;\mathbb{R}^n)$.

Let $\psi(\cdot)\in L^q_{\mathcal{F}_T}(0,T;\mathbb{R}^n)\subseteq L^2_{\mathcal{F}_T}(0,T;\mathbb{R}^n)$. According to Step 1, there exists a unique adapted M-solution $(Y(\cdot),Z(\cdot\,,\cdot))\in\mathcal{M}^2[0,T]$. We want to show that in the current case, $(Y(\cdot),Z(\cdot\,,\cdot))$

is actually in $\mathcal{M}^q[0,T]$. To show this, for the obtained adapted M-solution $(Y(\cdot),Z(\cdot\,,\cdot))$, let us consider the following MF-BSVIE:

$$\begin{aligned}\widetilde Y(t)=\psi(t)+&\int_t^T g(t,s,\widetilde Y(s),\widetilde Z(t,s),Z(s,t),\Gamma(t,s,\widetilde Y(s),\widetilde Z(t,s),Z(s,t))ds\\&-\int_t^T\widetilde Z(t,s)dW(s),\qquad\qquad t\in[0,T].\end{aligned}\tag{3.17}$$

For any $(y(\cdot),z(\cdot\,,\cdot))\in\mathcal{M}^p[0,T]$, by Proposition 3.1 (with $p=q$), the following MF-BSVIE admits a unique adapted M-solution $(\widetilde Y(\cdot),\widetilde Z(\cdot\,,\cdot))\in\mathcal{M}^q[0,T]$:

$$\begin{aligned}\widetilde Y(t)=\psi(t)+&\int_t^T g(t,s,y(s),\widetilde Z(t,s),Z(s,t),\Gamma(t,s,y(s),\widetilde Z(t,s),Z(s,t))ds\\&-\int_t^T\widetilde Z(t,s)dW(s),\qquad\qquad t\in[0,T].\end{aligned}\tag{3.18}$$

Thus, if we define $\widetilde\Theta(y(\cdot),z(\cdot\,,\cdot))=(\widetilde Y(\cdot),\widetilde Z(\cdot\,,\cdot))$, then $\widetilde\Theta:\mathcal{M}^q[0,T]\to\mathcal{M}^q[0,T]$. We now show that $\widetilde\Theta$ is a contraction on $\mathcal{M}^q[0,T]$ (Compare that Θ in Step 1 is a contraction on $\mathcal{M}^2[0,T]$). To this end, let $(y_i(\cdot),z_i(\cdot\,,\cdot))\in\mathcal{M}^q[0,T]$ and let

$$(\widetilde Y_i(\cdot),\widetilde Z_i(\cdot\,,\cdot))=\widetilde\Theta(y_i(\cdot),z_i(\cdot\,,\cdot)),\qquad i=1,2.$$

Then by Proposition 3.1 (with $p=q$), we have

$$\begin{aligned}&\mathbb{E}\Big[|\widetilde Y_1(t)-\widetilde Y_2(t)|^q+\Big(\int_t^T|\widetilde Z_1(t,s)-\widetilde Z_2(t,s)|^2ds\Big)^{\frac q2}\Big]\\&\le K\mathbb{E}\Big(\int_t^T|g(t,s,y_1(s),\widetilde Z_1(t,s),Z(s,t),\Gamma(t,s,y_1(s),\widetilde Z_1(t,s),Z(s,t)))\\&\qquad-g(t,s,y_2(s),\widetilde Z_1(t,s),Z(s,t),\Gamma(t,s,y_2(s),\widetilde Z_1(t,s),Z(s,t)))|ds\Big)^q\\&\le K\mathbb{E}\Big[\int_t^T\Big(|y_1(s)-y_2(s)|+|\Gamma(t,s,y_1(s),\widetilde Z_1(t,s),Z(s,t))-\Gamma(t,s,y_2(s),\widetilde Z_1(t,s),Z(s,t))|\Big)ds\Big]^q\\&\le K\mathbb{E}\Big[\int_t^T\Big(|y_1(s)-y_2(s)|+\mathbb{E}|y_1(s)-y_2(s)|\Big)ds\Big]^q\le K\mathbb{E}\int_t^T|y_1(s)-y_2(s)|^qds.\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T e^{\beta t}|\widetilde Y_1(t)-\widetilde Y_2(t)|^qdt+\int_0^T e^{\beta t}\Big(\int_t^T|\widetilde Z_1(t,s)-\widetilde Z_2(t,s)|^2ds\Big)^{\frac q2}dt\Big]\\&\le K\mathbb{E}\int_0^T e^{\beta t}\int_t^T|y_1(s)-y_2(s)|^qdsdt=K\mathbb{E}\int_0^T\int_0^s e^{\beta t}|y_1(s)-y_2(s)|^qdtds\\&\le\frac K\beta\int_0^T e^{\beta t}|y_1(t)-y_2(t)|^qdt.\end{aligned}$$

Hence, $\widetilde\Theta$ is a contraction on $\mathcal{M}^q[0,T]$ (with the equivalent norm). Hence, (3.18) admits a unique adapted M-solution $(\widetilde Y(\cdot),\widetilde Z(\cdot\,,\cdot))\in\mathcal{M}^q[0,T]\subseteq\mathcal{M}^2[0,T]$. Then by the uniqueness of adapted solutions in $\mathcal{M}^2[0,T]$ of (3.18), it is necessary that

$$(Y(\cdot),Z(\cdot\,,\cdot))=(\widetilde Y(\cdot),\widetilde Z(\cdot\,,\cdot))\in\mathcal{M}^q[0,T].$$

Step 3. Some estimates.

According to Proposition 3.1, we have

$$
\begin{aligned}
&\mathbb{E}\Big\{|Y(t)|^q+\Big(\int_t^T|Z(t,s)|^2ds\Big)^{\frac{q}{2}}\Big\}\\
&\leq K\mathbb{E}\Big\{|\psi(t)|^q+\Big(\int_t^T|g(t,s,Y(s),0,Z(s,t),\Gamma(t,s,Y(s),0,Z(s,t))|ds\Big)^q\Big\}\\
&\leq K\mathbb{E}\Big\{1+|\psi(t)|^q+\Big[\int_t^T\Big(|Y(s)|+|Z(s,t)|^{\frac{2}{q}}+|\Gamma(t,s,Y(s),0,Z(s,t))|\Big)ds\Big]^q\Big\}\\
&\leq K\mathbb{E}\Big\{1+|\psi(t)|^q+\Big[\int_t^T\Big(|Y(s)|+|Z(s,t)|^{\frac{2}{q}}\Big)ds\Big]^q\Big\}\\
&\leq K\mathbb{E}\Big\{1+|\psi(t)|^q+\int_t^T|Y(s)|^qds+\Big(\int_t^T|Z(s,t)|^{\frac{2}{q}}ds\Big)^q\Big\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathbb{E}\Big\{\int_0^T e^{\beta t}|Y(t)|^qdt+\int_0^T e^{\beta t}\Big(\int_t^T|Z(t,s)|^2ds\Big)^{\frac{q}{2}}dt\Big\}\\
&\leq K\mathbb{E}\Big\{\int_0^T e^{\beta t}\Big(1+|\psi(t)|^q\Big)dt+\int_0^T e^{\beta t}\int_t^T|Y(s)|^qdsdt+\int_0^T e^{\beta t}\Big(\int_t^T|Z(s,t)|^{\frac{2}{q}}ds\Big)^qdt\Big\}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\mathbb{E}\int_0^T e^{\beta t}\Big(\int_t^T|Z(s,t)|^{\frac{2}{q}}ds\Big)^qdt=\mathbb{E}\int_0^T e^{\beta t}\Big(\int_t^T e^{-\frac{\beta}{q}s}e^{\frac{\beta}{q}s}|Z(s,t)|^{\frac{2}{q}}ds\Big)^qdt\\
&\leq \mathbb{E}\int_0^T e^{\beta t}\Big(\int_t^T e^{-\frac{\beta}{q-1}s}ds\Big)^{q-1}\Big(\int_t^T e^{\beta s}|Z(s,t)|^2ds\Big)dt\\
&=\mathbb{E}\int_0^T e^{\beta t}\Big[\frac{q-1}{\beta}\Big(e^{\frac{-\beta t}{q-1}}-e^{-\frac{\beta T}{q-1}}\Big)\Big]^{q-1}\Big(\int_t^T e^{\beta s}|Z(s,t)|^2ds\Big)dt\\
&\leq\frac{(q-1)^{q-1}}{\beta^{q-1}}\mathbb{E}\int_0^T\int_0^s e^{\beta s}|Z(s,t)|^2dtds\leq\frac{(q-1)^{q-1}}{\beta^{q-1}}\mathbb{E}\int_0^T e^{\beta t}|Y(t)|^2dt.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\mathbb{E}\Big\{\int_0^T e^{\beta t}|Y(t)|^qdt+\int_0^T e^{\beta t}\Big(\int_t^T|Z(t,s)|^2ds\Big)^{\frac{q}{2}}dt\Big\}\\
&\leq K\mathbb{E}\Big\{\int_0^T e^{\beta t}\Big(1+|\psi(t)|^q\Big)dt+\frac{1}{\beta}\int_0^T e^{\beta t}|Y(t)|^qdt+\frac{1}{\beta^{q-1}}\int_0^T e^{\beta t}|Y(t)|^2dt\Big\}.
\end{aligned}
$$

By choosing $\beta>0$ large, we obtain

$$
\mathbb{E}\Big\{\int_0^T e^{\beta t}|Y(t)|^qdt+\int_0^T e^{\beta t}\Big(\int_t^T|Z(t,s)|^2ds\Big)^{\frac{q}{2}}dt\Big\}\leq K\mathbb{E}\Big(1+\int_0^T e^{\beta t}|\psi(t)|^qdt\Big).
$$

Then (3.12) follows.

Finally, let $\psi_i(\cdot) \in L^q_{\mathcal{F}_T}(0,T;\mathbb{R}^n)$, $g_i(\cdot)$ and $\theta_i(\cdot)$ satisfy (H3)$_q$. Observe that

$$
\begin{aligned}
Y_1(t) &= \psi_1(t) + \int_t^T g_1(t,s,Y_1(s),Z_1(t,s),Z_1(s,t),\Gamma_1(t,s,Y_1(s),Z_1(t,s),Z_1(s,t)))ds \\
&\qquad\qquad - \int_t^T Z_1(t,s)dW(s) \\
&= \psi_1(t) + \int_t^T g_1(t,s,Y_1(s),Z_1(t,s),Z_1(s,t),\Gamma_1(t,s,Y_1(s),Z_1(t,s),Z_1(s,t)))ds \\
&\qquad - \int_t^T g_2(t,s,Y_1(s),Z_1(t,s),Z_1(s,t),\Gamma_2(t,s,Y_1(s),Z_1(t,s),Z_1(s,t)))ds \\
&\qquad + \int_t^T g_2(t,s,Y_1(s),Z_1(t,s),Z_1(s,t),\Gamma_2(t,s,Y_1(s),Z_1(t,s),Z_1(s,t)))ds \\
&\qquad - \int_t^T g_2(t,s,Y_2(s),Z_1(t,s),Z_2(s,t),\Gamma_2(t,s,Y_2(s),Z_1(t,s),Z_2(s,t)))ds \\
&\qquad + \int_t^T g_2(t,s,Y_2(s),Z_1(t,s),Z_2(s,t),\Gamma_2(t,s,Y_2(s),Z_1(t,s),Z_2(s,t)))ds \\
&\qquad\qquad - \int_t^T Z_1(t,s)dW(s) \\
&\equiv \widehat{\psi}_1(t) + \int_t^T g_2(t,s,Y_2(s),Z_1(t,s),Z_2(s,t),\Gamma_2(t,s,Y_2(s),Z_1(t,s),Z_2(s,t)))ds \\
&\qquad\qquad - \int_t^T Z_1(t,s)dW(s),
\end{aligned}
$$

with $\widehat{\psi}_1(\cdot)$ defined in an obvious way. Then by Proposition 3.1 (with $p=2$), we obtain

$$
\begin{aligned}
&\mathbb{E}\Big[|Y_1(t)-Y_2(t)|^2 + \int_t^T |Z_1(t,s)-Z_2(t,s)|^2 ds\Big] \le K\mathbb{E}|\widehat{\psi}_1(t)-\psi_2(t)|^2 \\
&\le K\mathbb{E}\Big\{|\psi_1(t)-\psi_2(t)|^2 \\
&\qquad + \Big(\int_t^T |g_1(t,s,Y_1(s),Z_1(t,s),Z_1(s,t),\Gamma_1(t,s,Y_1(s),Z_1(t,s),Z_1(s,t))) \\
&\qquad\qquad - g_2(t,s,Y_1(s),Z_1(t,s),Z_1(s,t),\Gamma_2(t,s,Y_1(s),Z_1(t,s),Z_1(s,t)))|ds\Big)^2 \\
&\qquad + \Big(\int_t^T |g_2(t,s,Y_1(s),Z_1(t,s),Z_1(s,t),\Gamma_2(t,s,Y_1(s),Z_1(t,s),Z_1(s,t))) \\
&\qquad\qquad - g_2(t,s,Y_2(s),Z_1(t,s),Z_2(s,t),\Gamma_2(t,s,Y_2(s),Z_1(t,s),Z_2(s,t)))|ds\Big)^2\Big\} \\
&\le K\mathbb{E}\Big\{|\psi_1(t)-\psi_2(t)|^2 + \Big(\int_t^T |(g_1-g_2)(t,s)|ds\Big)^2 \\
&\qquad + \Big[\int_t^T \Big(|Y_1(s)-Y_2(s)| + |Z_1(s,t)-Z_2(s,t)|\Big)ds\Big]^2\Big\}.
\end{aligned}
$$

Then similar to the proof of the contraction for Θ, we can obtain our stability estimate (3.13). $\square$

Let us make some remarks on the above result, together with its proof.

First of all, we have seen that the growth of the maps

$$
\hat{z} \mapsto g(t,s,y,z,\hat{z}), \quad (\hat{z},\hat{z}') \mapsto \theta(t,s,y,z,\hat{z},y',z',\hat{z}') \tag{3.19}
$$

plays an important role in proving the well-posedness of MF-BSVIEs, especially for the case of $p > 2$. When $p \in (1, 2]$, the adapted M-solutions for BSVIEs was discussed in [35]. It is possible to adopt the idea of [35] to treat MF-BSVIEs for $p \in (1, 2)$. If (H3)$_\infty$ holds, then for any $p > 1$, as long as $\psi(\cdot) \in L^p_{\mathcal{F}_T}(0, T; \mathbb{R}^n)$, (3.10) admits a unique adapted M-solution $(Y(\cdot), Z(\cdot\,,\cdot)) \in \mathcal{M}^p[0, T]$. On the other hand, if the maps in (3.19) grow linearly, the adapted M-solution $(Y(\cdot), Z(\cdot\,,\cdot))$ of (3.10) may not be in $\mathcal{M}^p[0, T]$ for $p > 2$, even if $\psi(\cdot) \in L^p_{\mathcal{F}_T}(0, T; \mathbb{R}^n)$. This can be seen from the following example.

Example 3.3. Consider BSVIE:

$$Y(t) = \psi(t) + \int_t^T Z(s,t)ds - \int_t^T Z(t,s)dW(s), \quad t \in [0,T]. \tag{3.20}$$

Let

$$\psi(t) \equiv \int_0^T \psi_1(s)dW(s), \qquad \forall t \in [0,T],$$

with $\psi_1(\cdot)$ being deterministic and

$$\psi_1(\cdot) \in L^2(0,T) \setminus \bigcup_{p>2} L^p(0, T-\delta; \mathbb{R}),$$

for some fixed $\delta \in (0, T; \mathbb{R})$. Thus, for any $p > 1$,

$$\mathbb{E}\int_0^T |\psi(t)|^p ds = T\mathbb{E}\Big|\int_0^T \psi_1(s)dW(s)\Big|^p \le C\Big(\int_0^T |\psi_1(s)|^2 ds\Big)^{\frac{p}{2}},$$

which means $\psi(\cdot) \in L^p_{\mathcal{F}_T}(0, T; \mathbb{R})$ for any $p > 1$. If we define

$$\begin{cases} Y(t) = \displaystyle\int_0^t \psi_1(s)dW(s) + \psi_1(t)(T-t), & t \in [0,T], \\ Z(t,s) = \psi_1(s), \qquad (s,t) \in [0,T]^2, \end{cases}$$

then

$$\begin{aligned} Y(t) &= \int_0^t \psi_1(s)dW(s) + \psi_1(t)(T-t) \\ &= \psi(t) - \int_t^T \psi_1(s)dW(s) + \int_t^T \psi_1(t)ds \\ &= \psi(t) - \int_t^T Z(s,t)ds + \int_t^T Z(t,s)dW(s). \end{aligned}$$

This means that $(Y(\cdot), Z(\cdot\,,\cdot))$ is the adapted M-solution of (3.20). We claim that $Y(\cdot) \notin L^p_{\mathbb{F}}(0, T; \mathbb{R})$, for any $p > 2$. In fact, if $Y(\cdot) \in L^p_{\mathbb{F}}(0, T; \mathbb{R})$ for some $p > 2$, then

$$\begin{aligned} \delta^p \mathbb{E}\int_0^{T-\delta} |\psi_1(t)|^p dt &\le \int_0^T (T-t)^p |\psi_1(t)|^p dt \le 2^{p-1}\mathbb{E}\int_0^T \Big(|Y(t)|^p + \Big|\int_0^t \psi_1(s)dW(s)\Big|^p\Big)dt \\ &\le K\Big\{\int_0^T \mathbb{E}|Y(t)|^p dt + \Big(\int_0^T |\psi(s)|^2 ds\Big)^{\frac{p}{2}}\Big) < \infty. \end{aligned}$$

This is a contradiction.

Next, if

$$g_i(t,s,y,z,\hat z) = g_i(t,s,y,z), \qquad \theta_i(t,s,y,z,\hat z,y',z',\hat z') = \theta_i(t,s,y,z,y',z'),$$

then the stability estimate (3.13) can be improved to

$$
\begin{aligned}
&\|(Y_1(\cdot),Z_1(\cdot\,,\cdot))-(Y_2(\cdot),Z_2(\cdot\,,\cdot))\|^q_{\mathcal{M}^q[0,T]}\\
&\le K\mathbb{E}\Big\{\int_0^T|\psi_1(t)-\psi_2(t)|^q dt+\int_0^T\Big(\int_t^T|(g_1-g_2)(t,s)|ds\Big)^q dt\Big\},
\end{aligned}\tag{3.21}
$$

for any $q>2$.

We point out that even for the special case of BSVIEs, the proof we provided here significantly simplifies that given in [41]. The key is that we have a better understanding of the term $Z(s,t)$ in the drift, and find a new way to treat it (see (3.16)).

Now, let us look at linear MF-BSVIE (2.38). It is not hard to see that under (L2), we have (H3)$_q$ with $q=2$. Hence, we have the following corollary.

Corollary 3.4. Let (L2) hold. Then for any $\psi(\cdot)\in L^2_{\mathcal{F}_T}(0,T;\mathbb{R}^n)$, (2.38) admits a unique adapted M-solution $(Y(\cdot),Z(\cdot\,,\cdot))\in\mathcal{M}^2[0,T]$.

4 Duality Principles.

In this section, we are going to establish two duality principles between linear MF-FSVIEs and linear MF-BSVIEs. Let us first consider the following linear MF-FSVIE (2.32) which is rewritten below (for convenience):

$$
\begin{aligned}
X(t)=\varphi(t)&+\int_0^t\Big(A_0(t,s)X(s)+\mathbb{E}'\Big[C_0(t,s)X(s)\Big]\Big)ds\\
&+\int_0^t\Big(A_1(t,s)X(s)+\mathbb{E}'\Big[C_1(t,s)X(s)\Big]\Big)dW(s),\qquad t\in[0,T].
\end{aligned}\tag{4.1}
$$

Let (L1) hold and $\varphi(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$. Then by Corollary 2.7, (4.1) admits a unique solution $X(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$. Now, let $(Y(\cdot),Z(\cdot\,,\cdot))\in\mathcal{M}^2[0,T]$ be undetermined, and we observe the following:

$$
\begin{aligned}
\mathbb{E}\int_0^T\langle\,Y(t),\varphi(t)\,\rangle\,dt&=\mathbb{E}\int_0^T\langle\,Y(t),X(t)-\int_0^t\Big(A_0(t,s)X(s)+\mathbb{E}'[C_0(t,s)X(s)]\Big)ds\,\rangle\,dt\\
&-\mathbb{E}\int_0^T\langle\,Y(t),\int_0^t\Big(A_1(t,s)X(s)+\mathbb{E}'[C_1(t,s)X(s)]\Big)dW(s)\,\rangle\,dt\\
&\equiv\mathbb{E}\int_0^T\langle\,Y(t),X(t)\,\rangle\,dt-\sum_{i=1}^4 I_i.
\end{aligned}
$$

We now look at each term I_i. First, for I_1, we have

$$
I_1=\mathbb{E}\int_0^T\int_0^t\langle\,Y(t),A_0(t,s)X(s)\,\rangle\,dsdt=\mathbb{E}\int_0^T\langle\,X(t),\int_t^T A_0(s,t)^TY(s)ds\,\rangle\,dt.
$$

Next, for I_2, let us pay some extra attention on ω and ω',

$$
\begin{aligned}
I_2&=\mathbb{E}\int_0^T\!\!\int_0^t\langle\,Y(t),\mathbb{E}'[C_0(t,s)X(s)]\,\rangle\,dsdt=\mathbb{E}'\mathbb{E}\int_0^T\!\!\int_s^T\langle\,C_0(t,s,\omega,\omega')^TY(t,\omega),X(s,\omega')\,\rangle\,dtds\\
&=\mathbb{E}\mathbb{E}^*\!\int_0^T\!\!\int_s^T\langle\,C_0(t,s,\omega^*,\omega)^TY(t,\omega^*),X(s,\omega)\,\rangle\,dtds=\mathbb{E}\!\int_0^T\langle\,X(t),\int_t^T\mathbb{E}^*[C_0(s,t)^TY(s)]ds\,\rangle\,dt.
\end{aligned}
$$

Here, we have introduced the notation $\mathbb{E}^*$, whose definition is obvious from the above, to distinguish $\mathbb{E}$ (and $\mathbb{E}'$). For I_3, we have

$$\begin{aligned}
I_3 &= \mathbb{E}\int_0^T \langle\, Y(t), \int_0^t A_1(t,s)X(s)dW(s)\,\rangle\, dt \\
&= \mathbb{E}\int_0^T \langle\, \mathbb{E}Y(t) + \int_0^t Z(t,s)dW(s), \int_0^t A_1(t,s)X(s)dW(s)\,\rangle\, dt \\
&= \mathbb{E}\int_0^T \int_0^t \langle\, Z(t,s), A_1(t,s)X(s)\,\rangle\, dsdt = \mathbb{E}\int_0^T \langle\, X(t), \int_t^T A_1(s,t)^T Z(s,t)ds\,\rangle\, dt.
\end{aligned}$$

Finally, we look at I_4.

$$\begin{aligned}
I_4 &= \mathbb{E}\int_0^T \langle\, Y(t), \int_0^t \mathbb{E}'[C_1(t,s)X(s)]dW(s)\,\rangle\, dt \\
&= \mathbb{E}\int_0^T \int_0^t \langle\, Z(t,s), \mathbb{E}'[C_1(t,s)X(s)]\,\rangle\, dsdt \\
&= \mathbb{E}'\mathbb{E}\int_0^T \int_s^T \langle\, Z(t,s,\omega), C_1(t,s,\omega,\omega')X(s,\omega')\,\rangle\, dtds \\
&= \mathbb{E}\int_0^T \langle\, X(t), \int_t^T \mathbb{E}^*[C_1(s,t)^T Z(s,t)]ds\,\rangle\, dt.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\mathbb{E}\int_0^T \langle\, Y(t), \varphi(t)\,\rangle\, dt = \mathbb{E}\int_0^T \langle\, X(t), Y(t) - \int_t^T \Big(&A_0(s,t)^T Y(s) + A_1(s,t)^T Z(s,t) \\
&+\mathbb{E}^*\Big[C_0(s,t)^T Y(s) + C_1(s,t)^T Z(s,t)\Big]\Big)ds\,\rangle\, dt.
\end{aligned}$$

On the other hand, suppose (L1)$'$ holds and $\varphi(\cdot) \in C_{\mathbb{F}}^p([0,T];\mathbb{R}^n)$. Then $X(\cdot) \in C_{\mathbb{F}}^p([0,T];\mathbb{R}^n)$.

Consequently, we obtain the following *duality principle for MF-FSVIEs* whose proof is clear from the above.

Theorem 4.1. *Let* (L1) *hold, and* $\varphi(\cdot), \psi(\cdot) \in L_{\mathbb{F}}^2(0,T;\mathbb{R}^n)$. *Let* $X(\cdot) \in L_{\mathbb{F}}^2(0,T;\mathbb{R}^n)$ *be the solution to the linear MF-FSVIE* (4.1), *and* $(Y(\cdot), Z(\cdot\,,\cdot)) \in \mathcal{M}^2[0,T]$ *be the adapted M-solution to the following linear MF-BSVIE:*

$$\begin{aligned}
Y(t) = \psi(t) + \int_t^T \Big(&A_0(s,t)^T Y(s) + A_1(s,t)^T Z(s,t) \\
&+\mathbb{E}^*\Big[C_0(s,t)^T Y(s) + C_1(s,t)^T Z(s,t)\Big]\Big)ds - \int_t^T Z(t,s)dW(s).
\end{aligned} \tag{4.2}$$

Then

$$\mathbb{E}\int_0^T \langle\, X(t), \psi(t)\,\rangle\, dt = \mathbb{E}\int_0^T \langle\, Y(t), \varphi(t)\,\rangle\, dt. \tag{4.3}$$

We call (4.2) the adjoint equation of (4.1). The above duality principle will be used in establishing Pontryagin's type maximum principle for optimal controls of MF-FSVIEs.

Next, different from the above, we want to start from the followng linear MF-BSVIE:

$$\begin{aligned}
Y(t) = \psi(t) + \int_t^T &\Big(\bar{A}_0(t,s)Y(s) + \bar{C}_0(t,s)Z(s,t) + \mathbb{E}'[\bar{A}_1(t,s)Y(s) + \bar{C}_1(t,s)Z(s,t)]\Big)ds \\
&- \int_t^T Z(t,s)dW(s), \qquad\qquad t \in [0,T].
\end{aligned} \tag{4.4}$$

This is a special case of (2.38) in which

$$\bar{B}_0(t,s)=0, \qquad \bar{B}_1(t,s)=0.$$

Under (L2), by Corollary 3.4, for any $\psi(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, (4.4) admits a unique adapted M-solution $(Y(\cdot),Z(\cdot\,,\cdot))\in\mathcal{M}^2[0,T]$. We point out here that for each $t\in[0,T)$, the maps

$$s\mapsto\bar{C}_0(t,s), \quad s\mapsto\bar{C}_1(t,s)$$

are $\mathbb{F}$-progressively measurable and $\mathbb{F}^2$-progressively measurable on $[t,T]$, respectively. Now, we let a process $X(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ be undetermined, and make the following calculation:

$$\begin{aligned}
&\mathbb{E}\int_0^T\langle\,X(t),\psi(t)\,\rangle\,dt=\mathbb{E}\int_0^T\langle\,X(t),Y(t)-\int_t^T\Big(\bar{A}_0(t,s)Y(s)+\bar{C}_0(t,s)Z(s,t)\\
&\qquad+\mathbb{E}'[\bar{A}_1(t,s)Y(s)+\bar{C}_1(t,s)Z(s,t)]\Big)ds-\int_t^T Z(t,s)dW(s)\,\rangle\,dt\\
&=\mathbb{E}\int_0^T\langle\,X(t),Y(t)\,\rangle\,dt-\mathbb{E}\int_0^T\int_t^T\langle\,X(t),\bar{A}_0(t,s)Y(s)\,\rangle\,dsdt\\
&\quad-\mathbb{E}\int_0^T\int_t^T\langle\,X(t),\bar{C}_0(t,s)Z(s,t)\,\rangle\,dsdt-\mathbb{E}\int_0^T\int_t^T\langle\,X(t),\mathbb{E}'[\bar{A}_1(t,s)Y(s)]\,\rangle\,dsdt\\
&\quad-\mathbb{E}\int_0^T\int_t^T\langle\,X(t),\mathbb{E}'[\bar{C}_1(t,s)Z(s,t)]\,\rangle\,dsdt\equiv\mathbb{E}\int_0^T\langle\,X(t),Y(t)\,\rangle\,dt-\sum_{i=1}^4 I_i.
\end{aligned}$$

Similar to the above, we now look at the terms I_i $(i=1,2,3,4)$ one by one. First, we look at I_1:

$$\begin{aligned}
I_1&=\mathbb{E}\int_0^T\int_t^T\langle\,X(t),\bar{A}_0(t,s)Y(s)\,\rangle\,dsdt=\mathbb{E}\int_0^T\int_0^s\langle\,\bar{A}_0(t,s)^TX(t),Y(s)\,\rangle\,dtds\\
&=\mathbb{E}\int_0^T\langle\int_0^t\bar{A}_0(s,t)^TX(s)ds,Y(t)\,\rangle\,dt.
\end{aligned}$$

Next, for I_2, one has

$$\begin{aligned}
I_2&=\mathbb{E}\int_0^T\int_t^T\langle\,X(t),\bar{C}_0(t,s)Z(s,t)\,\rangle\,dsdt=\mathbb{E}\int_0^T\int_0^s\langle\,\bar{C}_0(t,s)^TX(t),Z(s,t)\,\rangle\,dtds\\
&=\int_0^T\mathbb{E}\int_0^t\langle\,\bar{C}_0(s,t)^TX(s),Z(t,s)\,\rangle\,dsdt\\
&=\int_0^T\mathbb{E}\,\langle\int_0^t\mathbb{E}[\bar{C}_0(s,t)^T\mid\mathcal{F}_s]X(s)dW(s),\int_0^t Z(t,s)dW(s)\,\rangle\,dt\\
&=\int_0^T\mathbb{E}\,\langle\int_0^t\mathbb{E}[\bar{C}_0(s,t)^T\mid\mathcal{F}_s]X(s)dW(s),Y(t)-\mathbb{E}Y(t)\,\rangle\,dt\\
&=\mathbb{E}\int_0^T\langle\int_0^t\mathbb{E}[\bar{C}_0(s,t)^T\mid\mathcal{F}_s]X(s)dW(s),Y(t)\,\rangle\,dt.
\end{aligned}$$

Now, for I_3,

$$\begin{aligned}
I_3&=\mathbb{E}\int_0^T\int_t^T\langle\,X(t),\mathbb{E}'[\bar{A}_1(t,s)Y(s)]\,\rangle\,dsdt=\mathbb{E}\mathbb{E}'\int_0^T\int_0^s\langle\,\bar{A}_1(t,s,\omega,\omega')^TX(t,\omega),Y(s,\omega')\,\rangle\,dtds\\
&=\mathbb{E}'\int_0^T\langle\int_0^t\mathbb{E}[\bar{A}_1(s,t,\omega,\omega')^TX(s,\omega)]ds,Y(t,\omega')\,\rangle\,dt\\
&=\mathbb{E}\int_0^T\langle\int_0^t\mathbb{E}^*[\bar{A}_1(s,t,\omega^*,\omega)^TX(s,\omega^*)]ds,Y(t,\omega)\,\rangle\,dt\\
&\equiv\mathbb{E}\int_0^T\langle\int_0^t\mathbb{E}^*[\bar{A}_1(s,t)^TX(s)]ds,Y(t)\,\rangle\,dt.
\end{aligned}$$

Finally, similar to the above, one has

$$
\begin{aligned}
I_4 &= \mathbb{E}\int_0^T \int_t^T \langle\, X(t), \mathbb{E}'[\bar{C}_1(t,s)Z(s,t)]\,\rangle\, dsdt \\
&= \mathbb{E}\mathbb{E}'\int_0^T \int_0^s \langle\, \bar{C}_1(t,s,\omega,\omega')^T X(t,\omega), Z(s,t,\omega')\,\rangle\, dtds \\
&= \mathbb{E}'\int_0^T \int_0^t \langle\, \mathbb{E}[\bar{C}_1(s,t,\omega,\omega')^T X(s,\omega)], Z(t,s,\omega')\,\rangle\, dsdt \\
&= \mathbb{E}\int_0^T \int_0^t \langle\, \mathbb{E}^*[\bar{C}_1(s,t)^T X(s)], Z(t,s)\,\rangle\, dsdt \\
&= \int_0^T \mathbb{E}\int_0^t \langle\, \mathbb{E}\Big[\mathbb{E}^*[\bar{C}_1(s,t)^T X(s)] \mid \mathcal{F}_s\Big], Z(t,s)\,\rangle\, dsdt \\
&= \int_0^T \mathbb{E}\,\langle \int_0^t \mathbb{E}^*\Big[\mathbb{E}[\bar{C}_1(s,t)^T \mid \mathcal{F}_s]X(s)\Big]dW(s), \int_0^t Z(t,s)dW(s)\,\rangle\, dt \\
&= \int_0^T \mathbb{E}\,\langle \int_0^t \mathbb{E}^*\Big[\mathbb{E}[\bar{C}_1(s,t)^T \mid \mathcal{F}_s]X(s)\Big]dW(s), Y(t)-\mathbb{E}Y(t)\,\rangle\, dt \\
&= \mathbb{E}\int_0^T \langle \int_0^t \mathbb{E}^*\Big[\mathbb{E}[\bar{C}_1(s,t)^T \mid \mathcal{F}_s]X(s)\Big]dW(s), Y(t)\,\rangle\, dt.
\end{aligned}
$$

Combining the above, we obtain

$$
\begin{aligned}
&\mathbb{E}\int_0^T \langle\, X(t),\psi(t)\,\rangle\, dt = \mathbb{E}\int_0^T \langle\, X(t), Y(t)\,\rangle\, dt - \sum_{i=1}^4 I_i \\
&= \mathbb{E}\int_0^T \langle\, Y(t), X(t) - \int_0^t \Big(\bar{A}_0(s,t)^T X(s) + \mathbb{E}^*[\bar{A}_1(s,t)^T X(s)]\Big)ds \\
&\qquad - \int_0^t \Big(\mathbb{E}[\bar{C}_0(s,t)^T \mid \mathcal{F}_s]X(s) + \mathbb{E}^*\Big[\mathbb{E}[\bar{C}_1(s,t)^T \mid \mathcal{F}_s]X(s)\Big]\Big)dW(s)\,\rangle\, dt.
\end{aligned}
\tag{4.5}
$$

Now, we are at the position to state and prove the following *duality principle for MF-BSVIEs*.

Theorem 4.2. *Let* (L2) *hold and* $\psi(\cdot) \in L^2_{\mathcal{F}_T}(0,T;\mathbb{R}^n)$. *Let* $(Y(\cdot), Z(\cdot\,,\cdot)) \in \mathcal{M}^2[0,T]$ *be the unique adapted M-solution of linear MF-BSVIE* (4.4). *Further, let* $\varphi(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ *and* $X(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ *be the solution to the following linear MF-FSVIE:*

$$
\begin{aligned}
X(t) = \varphi(t) &+ \int_0^t \Big(\bar{A}_0(s,t)^T X(s) + \mathbb{E}^*[\bar{A}_1(s,t)^T X(s)]\Big)ds \\
&+ \int_0^t \Big(\mathbb{E}[\bar{C}_0(s,t)^T \mid \mathcal{F}_s]X(s) + \mathbb{E}^*\Big[\mathbb{E}[\bar{C}_1(s,t)^T \mid \mathcal{F}_s]X(s)\Big]\Big)dW(s), \qquad t\in[0,T].
\end{aligned}
\tag{4.6}
$$

Then

$$
\mathbb{E}\int_0^T \langle\, Y(t),\varphi(t)\,\rangle\, dt = \mathbb{E}\int_0^T \langle\, X(t),\psi(t)\,\rangle\, dt. \tag{4.7}
$$

Proof. For linear MF-FSVIE (4.6), when (L2) holds, we have (L1). Hence, for any $\varphi(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, (4.6) admits a unique solution $X(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$. Then (4.7) follows from (4.5) immediately. $\square$

We call MF-FSVIE (4.6) the adjoint equation of MF-BSVIE (4.4). Such a duality principle will be used to establish comparison theorems for MF-BSVIEs. Note that since for $s<t$, $\bar{C}_0(s,t)^T$ is $\mathcal{F}_t$-measurable and not necessarily $\mathcal{F}_s$-measurable, we have

$$
\mathbb{E}[\bar{C}_0(s,t)^T \mid \mathcal{F}_s] \neq \bar{C}_0(s,t), \qquad t\in(s,T], \tag{4.8}
$$

in general. Likewise, in general,

$$\mathbb{E}[\bar{C}_1(s,t)^T \mid \mathcal{F}_s] \neq \bar{C}_1(s,t), \qquad t \in (s,T]. \tag{4.9}$$

We now make some comparison between Theorems 4.1 and 4.2.

First, we begin with linear MF-FSVIE (4.1) which is rewritten here for convenience:

$$\begin{aligned} X(t) = \varphi(t) &+ \int_0^t \Big(A_0(t,s)X(s) + \mathbb{E}'[C_0(t,s)X(s)]\Big)ds \\ &+ \int_0^t \Big(A_1(t,s)X(s) + \mathbb{E}'[C_1(t,s)X(s)]\Big)dW(s), \qquad t \in [0,T]. \end{aligned} \tag{4.10}$$

According to Theorem 4.1, the adjoint equation of (4.10) is MF-BSVIE (4.2). Now, we want to use Theorem 4.2 to find the adjoint equation of (4.2) which is regarded as (4.4) with

$$\begin{cases} \bar{A}_0(t,s) = A_0(s,t)^T, \qquad \bar{A}_1(t,s,\omega,\omega') = C_0(s,t,\omega',\omega)^T, \\ \bar{C}_0(t,s) = A_1(s,t)^T, \qquad \bar{C}_1(t,s,\omega,\omega') = C_1(s,t,\omega',\omega)^T. \end{cases}$$

Then, by Theorem 4.2, we obtain the adjoint equation (4.6) with the coefficients:

$$\begin{cases} \bar{A}_0(s,t)^T = A_0(t,s), \qquad \bar{A}_1(s,t,\omega',\omega)^T = C_0(t,s,\omega,\omega'), \\ \mathbb{E}[\bar{C}_0(s,t)^T \mid \mathcal{F}_s] = \mathbb{E}[A_1(t,s) \mid \mathcal{F}_s] = A_1(t,s), \\ \mathbb{E}[\bar{C}_1(s,t,\omega',\omega)^T \mid \mathcal{F}_s] = \mathbb{E}[C_1(t,s,\omega,\omega') \mid \mathcal{F}_s] = C_1(t,s,\omega,\omega'). \end{cases}$$

Hence, (4.10) is the adjoint equation of (4.2). Thus, we have the following conclusion:

Twice adjoint equation of a linear MF-FSVIE is itself.

Next, we begin with linear MF-BSVIE (4.4). From Theorem 4.2, we know that the adjoint equation is linear MF-FSVIE (4.6). Now, we want to use Theorem 4.1 to find the adjoint equation of (4.6) which is regarded as (4.10) with

$$\begin{cases} A_0(t,s) = \bar{A}_0(s,t)^T, \qquad C_0(t,s,\omega,\omega') = \bar{A}_1(s,t,\omega',\omega)^T, \\ A_1(t,s) = \mathbb{E}[\bar{C}_0(s,t)^T \mid \mathcal{F}_s], \qquad C_1(t,s,\omega,\omega') = \mathbb{E}[\bar{C}_1(s,t,\omega',\omega)^T \mid \mathcal{F}_s]. \end{cases}$$

Then by Theorem 4.2, the adjoint equation is given by (4.2) with coefficients:

$$\begin{cases} A_0(s,t)^T = \bar{A}_0(t,s), \qquad C_0(s,t,\omega',\omega)^T = \bar{A}_1(t,s,\omega,\omega'), \\ A_1(s,t)^T = \mathbb{E}[\bar{C}_0(t,s) \mid \mathcal{F}_t], \qquad C_1(s,t,\omega',\omega) = \mathbb{E}[\bar{C}_1(t,s,\omega,\omega') \mid \mathcal{F}_t]. \end{cases}$$

In another word, the twice adjoint equation of linear MF-BSVIE (4.4) is the following:

$$\begin{aligned} Y(t) = \psi(t) &+ \int_t^T \Big(\bar{A}_0(t,s)Y(s) + \mathbb{E}[\bar{C}_0(t,s) \mid \mathcal{F}_t]Z(s,t) \\ &+\mathbb{E}'\Big[\bar{A}_1(t,s)Y(s) + \mathbb{E}[\bar{C}_1(t,s) \mid \mathcal{F}_t]Z(s,t)\Big]\Big)ds - \int_t^T Z(t,s)dW(s), \quad t \in [0,T], \end{aligned} \tag{4.11}$$

which is different from (4.4), unless $\bar{C}_0(t,s)$ and $\bar{C}_1(t,s)$ are $\mathcal{F}_t$-measurable for all $(t,s) \in \Delta$. Thus, we have the following conclusion:

Twice adjoint of a linear MF-BSVIE is not necessarily itself.

5 Comparison Theorems.

In this section, we are going to establish some comparison theorems for MF-FSVIEs and MF-BSVIEs, allowing the dimension to be larger than 1. Let

$$\mathbb{R}^n_+ = \Big\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \geq 0,\ 1 \leq i \leq n\Big\}.$$

When $x \in \mathbb{R}^n_+$, we also denote it by $x \geq 0$, and say that x is *nonnegative.* By $x \leq 0$ and $x \geq y$ (if $x, y \in \mathbb{R}^n$), we mean $-x \geq 0$ and $x - y \geq 0$, respectively. Moreover, if $X(\cdot)$ is a process, then by $X(\cdot) \geq 0$, we mean

$$X(t) \geq 0, \qquad t \in [0, T], \quad \text{a.s.}$$

Also, $X(\cdot)$ is said to be *nondecreasing* if it is componentwise nondecreasing. Likewise, we may define $X(\cdot) \leq 0$ and $X(\cdot) \geq Y(\cdot)$ (if both $X(\cdot)$ and $Y(\cdot)$ are $\mathbb{R}^n$-valued processes), and so on.

In what follows, we let $e_i \in \mathbb{R}^n$ be the vector that the i-th entry is 1 and all other entries are zero. Also, we let

$$\left\{\begin{array}{l} \mathbb{M}^n_+ = \Big\{A = (a_{ij}) \in \mathbb{R}^{n\times n} \mid a_{ij} \geq 0,\ i \neq j\Big\} \equiv \Big\{A \in \mathbb{R}^{n\times n} \mid \langle\, Ae_i, e_j\,\rangle \geq 0,\ i \neq j\Big\}, \\ \widehat{\mathbb{M}}^{n\times m}_+ = \Big\{A = (a_{ij}) \in \mathbb{R}^{n\times m} \mid a_{ij} \geq 0,\ 1 \leq i \leq n,\ 1 \leq j \leq m\Big\}, \\ \mathbb{M}^n_0 = \Big\{A = (a_{ij}) \in \mathbb{R}^{n\times n} \mid a_{ij} = 0,\ i \neq j\Big\} \equiv \Big\{A \in \mathbb{R}^{n\times n} \mid \langle\, Ae_i, e_j\,\rangle = 0,\ i \neq j\Big\}. \end{array}\right.$$

Note that $\widehat{\mathbb{M}}^{n\times m}_+$ is the set of all $(n \times m)$ matrices with all the entries being nonnegative, $\mathbb{M}^n_+$ is the set of all $(n \times n)$ matrices with all the off-diagonal entries being nonnegative, and $\mathbb{M}^n_0$ is actually the set of all $(n \times n)$ diagonal matrices. Clearly, $\mathbb{M}^n_+$ and $\widehat{\mathbb{M}}^{n\times m}_+$ are closed convex cones of $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times m}$, respectively, and $\mathbb{M}^n_0$ is a proper subspace of $\mathbb{R}^{n\times n}$. Whereas, for $n = m = 1$, one has

$$\mathbb{M}^1_+ = \mathbb{M}^1_0 = \mathbb{R}, \qquad \widehat{\mathbb{M}}^{1\times 1}_+ = \mathbb{R}_+ \equiv [0, \infty). \tag{5.1}$$

We have the following simple result which will be useful below and whose proof is obvious.

Proposition 5.1. *Let $A \in \mathbb{R}^{n\times m}$. Then $A \in \widehat{\mathbb{M}}^{n\times m}_+$ if and only if*

$$Ax \geq 0, \qquad \forall x \in \mathbb{R}^m_+. \tag{5.2}$$

In what follows, we will denote $\widehat{\mathbb{M}}^n_+ = \widehat{\mathbb{M}}^{n\times n}_+$.

5.1 Comparison of solutions to MF-FSVIEs.

In this subsection, we would like to discuss comparison of solutions to linear MF-FSVIEs. There are some positive and also negative results. To begin with, let us first present an example of MF-FSDEs.

Example 5.2. Consider the following one-dimensional linear MF-FSDE, written in the integral form:

$$X(t) = 1 + \int_0^t \mathbb{E}X(s)dW(s), \qquad t \in [0, T].$$

Taking expectation, we have

$$\mathbb{E}X(t)=1, \qquad \forall t\in[0,T].$$

Consequently, the solution $X(\cdot)$ is given by

$$X(t)=1+\int_0^t dW(s)=1+W(t), \qquad t\in[0,T].$$

Thus, although $X(0)=1>0$, the following fails:

$$X(t)\geq 0, \qquad t\in[0,T], \quad \text{a.s.}$$

The above example shows that if the diffusion contains a nonlocal term in an MF-FSDE, we could not get an expected comparison of solutions, in general. Therefore, for linear MF-FSDEs, one had better only look at the following:

$$X(t)=x+\int_0^t\Big(A_0(s)X(s)+\mathbb{E}'[C_0(s)X(s)]\Big)ds+\int_0^t A_1(s)X(s)dW(s), \quad t\in[0,T], \tag{5.3}$$

with the diffusion does not contain a nonlocal term. For the above, we make the following assumption.

(C1) The maps

$$A_0,A_1:[0,T]\times\Omega\to\mathbb{R}^{n\times n}, \quad C_0:[0,T]\times\Omega^2\to\mathbb{R}^{n\times n},$$

are uniformly bounded, and they are $\mathbb{F}$-progressively measurable, and $\mathbb{F}^2$-progressively measurable, respectively.

Note that, due to (5.1), the above (C1) is always true if $n=1$. We now present the following comparison theorem for linear MF-FSDEs.

Proposition 5.3. *Let* (C1) *hold and moreover,*

$$A_0(s,\omega)\in\mathbb{M}_+^n, \quad C_0(s,\omega,\omega')\in\widehat{\mathbb{M}}_+^n, \quad A_1(s,\omega)\in\mathbb{M}_0^n, \qquad s\in[0,T], \ \text{ a.s. } \omega,\omega'\in\Omega. \tag{5.4}$$

Let $X(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ *be the solution of linear MF-FSDE* (5.3) *with* $x\geq 0$. *Then*

$$X(t)\geq 0, \qquad \forall t\in[0,T], \quad \text{a.s.} \tag{5.5}$$

Proof. It is known from Theorem 2.6 that as a special case of MF-FSVIE, the linear MF-FSDE (5.3) admits a unique solution $X(\cdot)\in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$ for any $x\in\mathbb{R}^n$, and any $p\geq 2$. Further, it is not hard to see that $X(\cdot)$ has continuous paths. Since the equation is linear, it suffices to show that $x\leq 0$ implies

$$X(t)\leq 0, \qquad t\in[0,T], \quad \text{a.s.} \tag{5.6}$$

To prove (5.6), we define a convex function

$$f(x)=\sum_{i=1}^n (x_i^+)^2, \qquad \forall x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n,$$

where $a^+ = \max\{a, 0\}$ for any $a \in \mathbb{R}$. Applying Itô's formula to $f(X(t))$, we get

$$f(X(t)) - f(x) = \int_0^t \Big[\langle f_x(X(s)), A_0(s)X(s) + \mathbb{E}'[C_0(s)X(s)] \rangle + \frac{1}{2} \langle f_{xx}(X(s))A_1(s)X(s), A_1(s)X(s) \rangle \Big] ds + \int_0^t \langle f_x(X(s)), A_1(s)X(s) \rangle \, dW(s).$$

We observe the following: (noting $A_0(s) \in \mathbb{M}_+^n$)

$$\begin{aligned}
\langle f_x(X(s)), A_0(s)X(s) \rangle &= \sum_{i,j=1}^n 2X_i(s)^+ \langle e_i, A_0(s)e_j \rangle X_j(s) \\
&= \sum_{i=1}^n 2X_i(s)^+ \langle e_i, A_0(s)e_i \rangle X_i(s) + \sum_{i \neq j} 2X_i(s)^+ \langle e_i, A_0(s)e_j \rangle X_j(s) \\
&\leq \sum_{i=1}^n 2[X_i(s)^+]^2 \langle e_i, A_0(s)e_i \rangle + \sum_{i \neq j} 2 \langle e_i, A_0(s)e_j \rangle X_i(s)^+ X_j(s)^+ \leq K f(X(s)).
\end{aligned}$$

Also, one has (making use of $C_0(s) \in \widehat{\mathbb{M}}_+^n$)

$$\begin{aligned}
&\mathbb{E} \langle f_x(X(s)), \mathbb{E}'[C_0(s)X(s)] \rangle \\
&= 2 \int_{\Omega^2} \sum_{i,j=1}^n X_i(s,\omega)^+ \langle e_i, C_0(s,\omega,\omega')e_j \rangle X_j(s,\omega') \mathbb{P}(d\omega)\mathbb{P}(d\omega') \\
&\leq 2 \int_{\Omega^2} \sum_{i,j=1}^n X_i(s,\omega)^+ \langle e_i, C_0(s,\omega,\omega')e_j \rangle X_j(s,\omega')^+ \mathbb{P}(d\omega)\mathbb{P}(d\omega') \\
&\leq K \Big(\mathbb{E}\Big[\sum_{i=1}^n X_i(s)^+ \Big] \Big)^2 \leq K \mathbb{E} f(X(s)).
\end{aligned}$$

Next, we have (noting $A_1(\cdot)$ and $f_{xx}(\cdot)$ are diagonal)

$$\begin{aligned}
&\frac{1}{2}\mathbb{E} \langle f_{xx}(X(s))A_1(s)X(s), A_1(s)X(s) \rangle = \frac{1}{2}\mathbb{E} \sum_{i=1}^n I_{(X_i(s) \geq 0)} \Big(\langle A_1(s)e_i, e_i \rangle X_i(s) \Big)^2 \\
&= \frac{1}{2}\mathbb{E} \sum_{i=1}^n \langle A_1(s)e_i, e_i \rangle^2 [X_i(s)^+]^2 \leq K f(X(s)).
\end{aligned}$$

Consequently,

$$\mathbb{E} f(X(t)) \leq f(x) + K \int_0^t \mathbb{E} f(X(s)) ds, \qquad t \in [0,T].$$

Hence, by Gronwall's inequality, we obtain

$$\sum_{i=1}^n \mathbb{E}|X_i(t)^+|^2 \leq K \sum_{i=1}^n |x_i^+|^2, \qquad t \in [0,T].$$

Therefore, if $x \leq 0$ (component-wise), then

$$\sum_{i=1}^n \mathbb{E}|X_i(t)^+|^2 = 0, \qquad \forall t \in [0,T].$$

This leads to (5.6). $\square$

We now make some observations on condition (5.4).

1. Let $C_0(\cdot)=0$, $A_1(\cdot)=0$, and $A_0(\cdot)$ be continuous and for some $i\neq j$,

$$\langle\, A_0(0)e_i, e_j\,\rangle<0,$$

i.e., at least one off-diagonal entry of $A_0(0)$ is negative. Then by letting $x=e_i$, we have

$$X_j(t)=\langle\, X(t),e_j\,\rangle=\int_0^t\langle\, A_0(s)X(s),e_j\,\rangle\,ds=\langle\, A_0(0)e_i,e_j\,\rangle\, t+o(t)<0,$$

for $t>0$ small. Thus, $X(0)\geq 0$ does not imply $X(t)\geq 0$.

2. Let $A_0(\cdot)=0$, $A_1(\cdot)=0$, and $C_0(\cdot)$ be continuous and for some $i\neq j$,

$$\langle\, C_0(0)e_i, e_j\,\rangle<0,$$

i.e., at least one off-diagonal entry of $C_0(0)$ is negative. Then by a similar argument as above, we have that $X(0)\geq 0$ does not imply $X(t)\geq 0$.

3. Let $A_0(\cdot)=0$, $C_0(\cdot)=0$ and for some $i\neq j$,

$$\int_0^T\mathbb{P}\Big(\langle\, A_1(s)e_i,e_j\,\rangle\neq 0\Big)ds>0,$$

i.e., at least one off-diagonal entry of $A_1(\cdot)$ is not identically zero. Then by letting $x=e_i$, we have

$$X_j(t)=\int_0^t\langle\, A_1(s)X(s),e_j\,\rangle\,dW(s)\not\equiv 0,\qquad t\in[0,T].$$

On the other hand,

$$\mathbb{E}X_j(t)=0,\qquad t\in[0,T].$$

Hence,

$$X_j(t)\geq 0,\qquad \forall t\in[0,T],\ \text{a.s.}$$

must fail.

4. Let $n=1$, $A_0(\cdot)=A_1(\cdot)=0$ and $C_0(\cdot)$ bounded, $\mathbb{F}$-adapted with

$$C_0(s)\neq 0,\quad \mathbb{E}C_0(s)=0,\qquad s\in[0,T].$$

This means that "$C_0(s)\geq 0$, $\forall s\in[0,T]$, a.s. " fails (or diagonal elements of $C_0(\cdot)$ are not all nonnegative). Consider the following MF-FSDE:

$$X(t)=1+\int_0^t C_0(s)\mathbb{E}X(s)ds,\qquad t\in[0,T].$$

Then

$$\mathbb{E}X(t)=1,\qquad t\in[0,T].$$

Hence,

$$X(t)=1+\int_0^t C_0(s)ds,\qquad t\in[0,T].$$

It is easy to choose a $C_0(\cdot)$ such that

$$X(t) \geq 0, \qquad \forall t \in [0, T], \text{ a.s.}$$

is violated.

The above observations show that, in some sense, conditions assumed in (5.4) are sharp for Proposition 5.3.

Based on the above, let us now consider the following linear MF-FSVIE:

$$\begin{aligned} X(t) = \varphi(t) + \int_0^t \Big(A_0(t,s)X(s) + \mathbb{E}'\Big[C_0(t,s)X(s)\Big]\Big)ds \\ + \int_0^t A_1(s)X(s)dW(s), \qquad\qquad t \in [0,T]. \end{aligned} \tag{5.7}$$

Note that $A_1(\cdot)$ is independent of t here. According to [34], we know that for (linear) FSVIEs (without the nonlocal term, i.e., $C_0(\cdot\,,\cdot) = 0$ in (5.7)), if the diffusion depends on both (t, s) and $X(\cdot)$, i.e., $A_1(t, s)$ really depends on (t, s), a comparison theorem will fail in general. Next, let us look at an example which is concerned with the free term $\varphi(\cdot)$.

Example 5.4. Consider the following one-dimensional FSVIE:

$$X(t) = T - t + \int_0^t bX(s)ds + \int_0^t \sigma X(s)dW(s), \qquad t \in [0, T],$$

for some $b, \sigma \in \mathbb{R}$. The above is equivalent to the following:

$$\begin{cases} dX(t) = [bX(t) - 1]dt + \sigma X(t)dW(t), \qquad t \in [0,T], \\ X(0) = T. \end{cases}$$

The solution to the above is explicitly given by the following:

$$X(t) = e^{(b-\frac{\sigma^2}{2})t+\sigma W(t)}\Big[T - \int_0^t e^{-(b-\frac{\sigma^2}{2})s-\sigma W(s)}ds\Big], \qquad t \in [0,T].$$

We know that as long as $\sigma \neq 0$, for any $t > 0$ small and any $K > 0$,

$$\mathbb{P}\Big(\int_0^t e^{-(b-\frac{\sigma^2}{2})s-\sigma W(s)}ds \geq K\Big) > 0.$$

Therefore, we must have

$$\mathbb{P}(X(t) < 0) > 0, \qquad \forall t > 0 \text{ (small)}.$$

On the other hand, if $\sigma = 0$, then

$$X(t) = e^{bt}\Big[T - \int_0^t e^{-bs}ds\Big], \qquad t \in [0,T].$$

Thus, when $b = 0$, one has

$$X(t) = T - t, \qquad t \in [0, T],$$

and when $b \neq 0$,

$$X(t) = e^{bt}T + \frac{1}{b}(1 - e^{bt}) = \frac{e^{bt}}{b}\Big(e^{-bt} - 1 + bT\Big), \qquad t \in [0,T].$$

Since

$$e^{\lambda} - 1 - \lambda > 0, \qquad \forall \lambda \neq 0,$$

we have that $b < 0$ implies

$$X(T) < 0.$$

The above example tells us that when $\sigma \neq 0$, or $\sigma = 0$ and $b < 0$, although the free term $\varphi(t) = T - t$ is nonnegative on $[0, T]$, the solution $X(\cdot)$ of the FSVIE (5.7) does not necessarily remain nonnegative on $[0, T]$. Consequently, nonnegativity of the free term is not enough for the solution of the MF-FSVIE to be nonnegative. Thus, besides the nonnegativity of the free term, some additional conditions are needed.

To present positive results, we introduce the following assumption.

(C2) The maps

$$A_0 : \Delta^* \times \Omega \to \mathbb{R}^{n\times n}, \quad A_1 : [0, T] \times \Omega \to \mathbb{R}^{n\times n}, \quad C_0 : \Delta^* \times \Omega^2 \to \mathbb{R}^{n\times n},$$

are measurable and uniformly bounded. For any $t \in [0, T]$, $s \mapsto (A_0(t, s), A_1(s))$ is $\mathbb{F}$-progressively measurable on $[0, t]$, and $s \mapsto C_0(t, s)$ is $\mathbb{F}^2$-progressively measurable on $[0, t]$.

We now present the following result which is simple but will be useful later.

Proposition 5.5. *Let* (C2) *hold. Further,*

$$A_0(t, s, \omega), C_0(t, s, \omega, \omega') \in \widehat{\mathbb{M}}^n_+, \quad A_1(s, \omega) = 0, \quad \text{a.e. } (t, s) \in \Delta^*, \text{ a.s. } \omega, \omega' \in \Omega. \tag{5.8}$$

Let $X(\cdot)$ be the solution to (5.7), *with $\varphi(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$ and $\varphi(\cdot) \geq 0$. Then*

$$X(t) \geq \varphi(t) \geq 0, \qquad t \in [0, T]. \tag{5.9}$$

Proof. Define

$$(\mathcal{A}X)(t) = \int_0^t \Big(A_0(t, s)X(s) + \mathbb{E}'[C_0(t, s)X(s)] \Big) ds, \qquad t \in [0, T].$$

By our condition, we see that

$$(\mathcal{A}X)(\cdot) \geq 0, \qquad \forall X(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n),\ X(\cdot) \geq 0.$$

Now, we define the following sequence

$$\begin{cases} X_0(\cdot) = \varphi(\cdot), \\ X_k(\cdot) = \varphi(\cdot) + (\mathcal{A}X_{k-1})(\cdot), \qquad k \geq 1. \end{cases}$$

It is easy to see that

$$X_k(\cdot) \geq \varphi(\cdot), \qquad \forall k \geq 0,$$

and

$$\lim_{k\to\infty} \|X_k(\cdot) - X(\cdot)\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)} = 0,$$

with $X(\cdot)$ being the solution to (5.7). Then it is easy to see that (5.9) holds. $\square$

For the case that the diffusion is nonzero in the equation, we have the following result.

Proposition 5.6. *Let* (C2) *hold. Suppose*

$$\begin{aligned} A_0(t,s,\omega)\in\mathbb{M}^n_+,\quad C_0(t,s,\omega,\omega')\in\widehat{\mathbb{M}}^n_+,\quad A_1(s,\omega)\in\mathbb{M}^n_0,\\ \text{a.e. } (t,s)\in\Delta^*,\ \text{a.s. } \omega,\omega'\in\Omega. \end{aligned}\tag{5.10}$$

Moreover, let $t\mapsto(\varphi(t),A_0(t,s),C_0(t,s))$ *be continuous, and* $\varphi(\cdot)\in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$ *for some* $p>2$. *Further,*

$$\begin{aligned} \varphi(t_1)\ge\varphi(t_0)\ge0,\quad A_0(t_1,s)\widehat{x}\ge A_0(t_0,s)\widehat{x},\quad C_0(t_1,s)\widehat{x}\ge C_0(t_0,s)\widehat{x},\\ \forall\, s\le t_0<t_1\le T,\ s\in[0,T],\ \widehat{x}\in\mathbb{R}^n_+,\ \text{a.s.} \end{aligned}\tag{5.11}$$

Let $X(\cdot)$ *be the solution of linear MF-FSVIE* (5.7). *Then*

$$X(t)\ge0,\qquad t\in[0,T],\ \text{a.s.}\tag{5.12}$$

Proof. Let $\Pi=\{\tau_k,0\le k\le N\}$ be an arbitrary set of finitely many $\mathbb{F}$-stopping times with $0=\tau_0<\tau_1<\cdots<\tau_N=T$, and we define its mesh size by

$$\|\Pi\|=\operatorname*{esssup}_{\omega\in\Omega}\max_{1\le k\le N}|\tau_k-\tau_{k-1}|.$$

Let

$$\begin{cases} A_0^\Pi(t,s)=\displaystyle\sum_{k=0}^{N-1}A_0(\tau_k,s)I_{[\tau_k,\tau_{k+1})}(t),\qquad C_0^\Pi(t,s)=\sum_{k=0}^{N-1}C_0(\tau_k,s)I_{[\tau_k,\tau_{k+1})}(t),\\ \varphi^\Pi(t)=\displaystyle\sum_{k=0}^{N-1}\varphi(\tau_k)I_{[\tau_k,\tau_{k+1})}(t). \end{cases}$$

Clearly, each $A_0(\tau_k,\cdot)$ is an $\mathbb{F}$-adapted bounded process, each $C_0(\tau_k,\cdot)$ is an $\mathbb{F}^2$-adapted bounded process, and each $\varphi(\tau_k)$ is an $\mathcal{F}_{\tau_k}$-measurable random variable. Moreover, for each $k\ge0$,

$$A_0(\tau_k,s)\in\mathbb{M}^n_+,\quad C_0(\tau_k,s)\in\widehat{\mathbb{M}}^n_+,\qquad s\in[\tau_k,\tau_{k+1}],\ \ \text{a.s. },\tag{5.13}$$

and

$$0\le\varphi(\tau_k)\le\varphi(\tau_{k+1}),\qquad\text{a.s.}\tag{5.14}$$

Now, we let $X^\Pi(\cdot)$ be the solution to the following MF-FSVIE:

$$\begin{aligned} X^\Pi(t)=\varphi^\Pi(t)+\int_0^t\Big(A_0^\Pi(t,s)X^\Pi(s)+\mathbb{E}'\Big[C_0^\Pi(t,s)X^\Pi(s)\Big]\Big)ds\\ +\int_0^tA_1(s)X^\Pi(s)dW(s),\qquad t\in[0,T]. \end{aligned}\tag{5.15}$$

Then on interval $[0,\tau_1)$, we have

$$X^\Pi(t)=\varphi(0)+\int_0^t\Big(A_0(0,s)X^\Pi(s)+\mathbb{E}'\Big[C_0(0,s)X^\Pi(s)\Big]\Big)ds+\int_0^tA_1(s)X^\Pi(s)dW(s),$$

which is an MF-FSDE, and $X^\Pi(\cdot)$ has continuous paths. From Proposition 5.3, we have

$$X^\Pi(t)\ge0,\qquad t\in[0,\tau_1),\ \text{a.s.}$$

In particular,

$$X^\Pi(\tau_1-0)=\varphi(0)+\int_0^{\tau_1}\Big(A_0(0,s)X^\Pi(s)+\mathbb{E}'\Big[C_0(0,s)X^\Pi(s)\Big]\Big)ds \\ +\int_0^{\tau_1}A_1(s)X^\Pi(s)dW(s)\geq 0. \tag{5.16}$$

Next, on $[\tau_1,\tau_2)$, we have (making use the above)

$$\begin{aligned}X^\Pi(t)&=\varphi(\tau_1)+\int_0^{\tau_1}\Big(A_0(\tau_1,s)X^\Pi(s)+\mathbb{E}'\Big[C_0(\tau_1,s)X^\Pi(s)\Big]\Big)ds+\int_0^{\tau_1}A_1(s)X^\Pi(s)dW(s)\\&\quad+\int_{\tau_1}^t\Big(A_0(\tau_1,s)X^\Pi(s)+\mathbb{E}'\Big[C_0(\tau_1,s)X^\Pi(s)\Big]\Big)ds+\int_{\tau_1}^tA_1(s)X^\Pi(s)dW(s)\\&=\varphi(\tau_1)-\varphi(0)+X^\Pi(\tau_1-0)\\&\quad+\int_0^{\tau_1}\Big\{\Big(A_0(\tau_1,s)-A_0(0,s)\Big)X^\Pi(s)+\mathbb{E}'\Big[\Big(C_0(\tau_1,s)-C_0(0,s)\Big)X^\Pi(s)\Big]\Big\}ds\\&\quad+\int_{\tau_1}^t\Big(A_0(\tau_1,s)X^\Pi(s)+\mathbb{E}'\Big[C_0(\tau_1,s)X^\Pi(s)\Big]\Big)ds+\int_{\tau_1}^tA_1(s)X^\Pi(s)dW(s)\\&\equiv\widetilde X(\tau_1)+\int_{\tau_1}^t\Big(A_0(\tau_1,s)X^\Pi(s)+\mathbb{E}'\Big[C_0(\tau_1,s)X^\Pi(s)\Big]\Big)ds+\int_{\tau_1}^tA_1(s)X^\Pi(s)dW(s),\end{aligned}$$

where, by our conditions assumed in (5.11), and noting (5.16),

$$\begin{aligned}\widetilde X(\tau_1)&\equiv\varphi(\tau_1)-\varphi(0)+X^\Pi(\tau_1-0)\\&\quad+\int_0^{\tau_1}\Big\{\Big(A_0(\tau_1,s)-A_0(0,s)\Big)X^\Pi(s)+\mathbb{E}'\Big[\Big(C_0(\tau_1,s)-C_0(0,s)\Big)X^\Pi(s)\Big]\Big\}ds\geq 0.\end{aligned}$$

Hence, by Proposition 5.3 again, one obtains

$$X^\Pi(t)\geq 0,\qquad t\in[\tau_1,\tau_2).$$

By induction, we see that

$$X^\Pi(t)\geq 0,\qquad t\in[0,T],\ \text{a.s.}$$

On the other hand, it is ready to show that

$$\lim_{\|\Pi\|\to0}\|X^\Pi(\cdot)-X(\cdot)\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)}=0,$$

Then (5.12) follows from the stability estimate in Corollary 2.7. □

We now look at the following (nonlinear) MF-FSVIEs with $i=0,1$:

$$X_i(t)=\varphi_i(t)+\int_0^tb_i(t,s,X_i(t),\Gamma_i^b(t,s,X_i(s)))ds+\int_0^t\sigma(s,X_i(s))dW(s),\quad t\in[0,T], \tag{5.17}$$

where

$$\Gamma_i^b(t,s,X_i(s))=\int_\Omega\theta_i^b(t,s,\omega,\omega',X_i(s,\omega),X_i(s,\omega'))\mathbb{P}(d\omega'). \tag{5.18}$$

Note that $\sigma(\cdot)$ does not contain a nonlocal term, and it is independent of $t\in[0,T]$, as well as $i=0,1$. The following result can be regarded as an extension of [34] from FSVIEs to MF-FSVIEs.

Theorem 5.7. *For $i = 0, 1$, let $b_i(\cdot), \sigma(\cdot), \theta_i^b(\cdot)$ appeared in* (5.17) *satisfy* (H1)–(H2) *and $\varphi_i(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$. Further, for all $x, \bar{x}, x' \in \mathbb{R}^n$, $\widehat{x} \in \mathbb{R}^n_+$, $\gamma \in \mathbb{R}^{m_1}$, almost all $(t,s) \in \Delta^*$ and almost sure $\omega, \omega' \in \Omega$,*

$$(b_0)_\gamma(t,s,\omega,x,\gamma) \in \widehat{\mathbb{M}}_+^{n\times m_1}, \qquad \sigma_x(s,\omega,x) \in \mathbb{M}_0^n, \tag{5.19}$$

and maps

$$\begin{aligned}
&t \mapsto (b_0)_x(t,s,\omega,x,\gamma)\widehat{x},\\
&t \mapsto (b_0)_\gamma(t,s,\omega,x,\gamma)(\theta_0^b)_x(t,s,\omega,\bar{x},x')\widehat{x},\\
&t \mapsto (b_0)_\gamma(t,s,\omega,x,\gamma)(\theta_0^b)_{x'}(t,s,\omega,\bar{x},x')\widehat{x},\\
&t \mapsto b_1(t,s,\omega,x,\gamma) - b_0(t,s,\omega,x,\gamma),\\
&t \mapsto \theta_1^b(t,s,\omega,\omega',x,x') - \theta_0^b(t,s,\omega,\omega',x,x'),\\
&t \mapsto (b_0)_\gamma(t,s,\omega,x,\gamma)\Big[\theta_1^b(t,s,\omega,\omega',x,x') - \theta_0^b(t,s,\omega,\omega',x,x')\Big],\\
&t \mapsto \varphi_1(t) - \varphi_0(t)
\end{aligned} \tag{5.20}$$

are continuous, nonnegative and nondecreasing on $[s,T]$. Let $X_i(\cdot) \in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$ be the solutions to the corresponding equation (5.17)*. Then*

$$X_0(t) \le X_1(t), \qquad \forall t \in [0,T], \text{ a.s.} \tag{5.21}$$

Proof. From the equations satisfied by $X_0(\cdot)$ and $X_1(\cdot)$, we have the following:

$$\begin{aligned}
X_1(t) - X_0(t) &= \varphi_1(t) - \varphi_0(t)\\
&\quad + \int_0^t \Big[b_1(t,s,X_1(s),\Gamma_1^b(t,s,X_1(s))) - b_0(t,s,X_0(s),\Gamma_0^b(t,s,X_0(s)))\Big]ds\\
&\quad + \int_0^t \Big[\sigma(s,X_1(s)) - \sigma(s,X_0(s))\Big]dW(s)\\
&= \widehat{\varphi}_1(t) - \widehat{\varphi}_0(t)\\
&\quad + \int_0^t \Big[b_0(t,s,X_1(s),\Gamma_0^b(t,s,X_1(s))) - b_0(t,s,X_0(s),\Gamma_0^b(t,s,X_0(s)))\Big]ds\\
&\quad + \int_0^t \Big[\sigma(s,X_1(s)) - \sigma(s,X_0(s))\Big]dW(s),
\end{aligned}$$

where (making use of Proposition 5.1 and (5.19)–(5.20))

$$\begin{aligned}
&\widehat{\varphi}_1(t) - \widehat{\varphi}_0(t) = \varphi_1(t) - \varphi_0(t)\\
&\qquad + \int_0^t \Big[b_1(t,s,X_1(s),\Gamma_1^b(t,s,X_1(s))) - b_0(t,s,X_1(s),\Gamma_0^b(t,s,X_1(s)))\Big]ds\\
&= \varphi_1(t) - \varphi_0(t) + \int_0^t \Big[b_1(t,s,X_1(s),\Gamma_1^b(t,s,X_1(s))) - b_0(t,s,X_1(s),\Gamma_1^b(t,s,X_1(s)))\Big]ds\\
&\qquad + \int_0^t \Big[\int_0^1 (b_0)_\gamma(t,s,X_1(s),\widetilde{\Gamma}_\lambda^b(t,s))d\lambda\Big]\Big(\Gamma_1^b(t,s,X_1(s))) - \Gamma_0^b(t,s,X_1(s)))\Big)ds \ge 0,
\end{aligned}$$

and nondecreasing in t, where

$$\widetilde{\Gamma}_\lambda^b(t,s) = (1-\lambda)\Gamma_0^b(t,s,X_1(s)) + \lambda\Gamma_1^b(t,s,X_1(s)).$$

Now, we look at the following:

$$\begin{aligned}
&b_0(t,s,X_1(s),\Gamma_0^b(t,s,X_1(s)))-b_0(t,s,X_0(s),\Gamma_0^b(t,s,X_0(s)))\\
&=\Big[\int_0^1(b_0)_x(t,s,X_\lambda(s),\Gamma_\lambda^b(t,s))d\lambda\Big]\Big(X_1(s)-X_0(s)\Big)\\
&\qquad+\Big[\int_0^1(b_0)_\gamma(t,s,X_\lambda(s),\Gamma_\lambda^b(t,s))d\lambda\Big]\Big(\Gamma_0^b(t,s,X_1(s))-\Gamma_0^b(t,s,X_0(s))\Big)\\
&\equiv(b_0)_x(t,s)\Big(X_1(s)-X_0(s)\Big)+(b_0)_\gamma(t,s)\Big(\Gamma_0^b(t,s,X_1(s))-\Gamma_0^b(t,s,X_0(s))\Big),
\end{aligned}$$

where

$$\left\{\begin{array}{l}X_\lambda(s)=(1-\lambda)X_0(s)+\lambda X_1(s),\\ \Gamma_\lambda^b(t,s)=(1-\lambda)\Gamma_0^b(t,s,X_0(s))+\lambda\Gamma_0^b(t,s,X_1(s)).\end{array}\right. \tag{5.22}$$

and

$$\left\{\begin{array}{l}(b_0)_x(t,s)=\displaystyle\int_0^1(b_0)_x(t,s,X_\lambda(s),\Gamma_\lambda^b(t,s))d\lambda,\\ (b_0)_\gamma(t,s)=\displaystyle\int_0^1(b_0)_\gamma(t,s,X_\lambda(s),\Gamma_\lambda^b(t,s))d\lambda.\end{array}\right.$$

Moreover,

$$\begin{aligned}
&\Gamma_0^b(t,s,X_1(s))-\Gamma_0^b(t,s,X_0(s))\\
&=\int_\Omega\Big[\theta_0^b(t,s,\omega,\omega',X_1(s,\omega),X_1(s,\omega'))-\theta_0^b(t,s,\omega,\omega',X_0(s,\omega),X_0(s,\omega'))\Big]\mathbb{P}(d\omega')\\
&=\Big\{\int_\Omega\Big[\int_0^1(\theta_0^b)_x(t,s,\omega,\omega',X_\lambda(s,\omega),X_\lambda(s,\omega'))d\lambda\Big]\mathbb{P}(d\omega')\Big\}\Big(X_1(s,\omega)-X_0(s,\omega)\Big)\\
&\qquad+\int_\Omega\Big[\int_0^1(\theta_0^b)_{x'}(t,s,\omega,\omega',X_\lambda(s,\omega),X_\lambda(s,\omega'))d\lambda\Big]\Big(X_1(s,\omega')-X_0(s,\omega')\Big)\mathbb{P}(d\omega')\\
&=\mathbb{E}'\Big[(\theta_0^b)_x(t,s)\Big]\Big(X_1(s)-X_0(s)\Big)+\mathbb{E}'\Big[(\theta_0^b)_{x'}(t,s)\Big(X_1(s,\omega')-X_0(s,\omega')\Big)\Big],
\end{aligned}$$

where

$$\left\{\begin{array}{l}(\theta_0^b)_x(t,s)=\displaystyle\int_0^1(\theta_0^b)_x(t,s,\omega,\omega',X_\lambda(s,\omega),X_\lambda(s,\omega'))d\lambda,\\ (\theta_0^b)_{x'}(t,s)=\displaystyle\int_0^1(\theta_0^b)_{x'}(t,s,\omega,\omega',X_\lambda(s,\omega),X_\lambda(s,\omega'))d\lambda,\end{array}\right. \tag{5.23}$$

and $X_\lambda(\cdot)$ is defined as (5.22). Thus,

$$\begin{aligned}
&b_0(t,s,X_1(s),\Gamma_0^b(t,s,X_1(s)))-b_0(t,s,X_0(s),\Gamma_0^b(t,s,X_0(s)))\\
&=\Big\{(b_0)_x(t,s)+\mathbb{E}'\Big[(b_0)_\gamma(t,s)(\theta_0^b)_x(t,s)\Big]\Big\}\Big(X_1(s)-X_0(s)\Big)\\
&\qquad+\mathbb{E}'\Big[(b_0)_\gamma(t,s)(\theta_0^b)_{x'}(t,s)\Big(X_1(s,\omega')-X_0(s,\omega')\Big)\Big]\\
&\equiv A_0(t,s)\Big(X_1(s)-X_0(s)\Big)+\mathbb{E}'\Big[C_0(t,s)\Big(X_1(s)-X_0(s)\Big)\Big],
\end{aligned}$$

where

$$\left\{\begin{array}{l}A_0(t,s)=(b_0)_x(t,s)+\mathbb{E}'\Big[(b_0)_\gamma(t,s)(\theta_0^b)_x(t,s)\Big]\in\mathbb{M}_+^n,\\ C_0(t,s)=(b_0)_\gamma(t,s)(\theta_0^b)_{x'}(t,s)\in\widehat{\mathbb{M}}_+^n,\end{array}\right.\qquad(t,s)\in\Delta^*,\ \text{a.s.} \tag{5.24}$$

Similarly,

$$\sigma(s,X_1(s))-\sigma(s,X_0(s))\equiv A_1(s)\Big(X_1(s)-X_0(s)\Big),$$

where

$$A_1(s)\equiv\int_0^1\sigma_x(s,X_\lambda(s))d\lambda\in\mathbb{M}_0^n,\qquad(t,s)\in\Delta^*,\ \text{a.s.}\tag{5.25}$$

Then we have

$$\begin{aligned}X_1(t)-X_0(t)=\widehat\varphi_1(t)-\widehat\varphi_0(t)+\int_0^t\Big\{A_0(t,s)\Big(X_1(s)-X_0(s)\Big)\\+\mathbb{E}'\Big[C_0(t,s)\Big(X_1(s)-X_0(s)\Big)\Big]\Big\}ds+\int_0^tA_1(s)\Big(X_1(s)-X_0(s)\Big)dW(s).\end{aligned}$$

From (5.19)–(5.20), we see that the coefficients of the above linear MF-FSVIE satisfy (C2), and $\widehat\varphi_1(\cdot)-\widehat\varphi_0(\cdot)$ is nonnegative and nondecreasing. Then (5.21) follows from Proposition 5.6. $\square$

From the above proof, we see that one may replace $b_0(\cdot)$ in conditions (5.19) by $b_1(\cdot)$. Also, by an approximation argument, we may replace the derivatives in (5.19) of $b_0(\cdot)$ and $\sigma(\cdot)$ by the corresponding difference quotients.

5.2 Comparison theorems for MF-BSVIEs.

In this subsection, we discuss comparison property for MF-BSVIEs. First, we consider the following linear MF-BSVIE:

$$\begin{aligned}Y(t)=\psi(t)+\int_t^T\Big(\bar A_0(t,s)Y(s)+\bar C_0(t)Z(s,t)+\mathbb{E}'\Big[\bar A_1(t,s)Y(s)\Big]\Big)ds\\-\int_t^TZ(t,s)dW(s),\qquad t\in[0,T].\end{aligned}\tag{5.26}$$

Note that $Z(t,s)$ does not appear in the whole drift term, and $Z(s,t)$ does not appear in the nonlocal term. Further, the coefficient of $Z(s,t)$ is independent of s. Let us introduce the following assumption.

(C3) The maps

$$\bar A_0:\Delta\times\Omega\to\mathbb{R}^{n\times n},\quad\bar C_0:[0,T]\times\Omega\to\mathbb{R}^{n\times n},\quad\bar A_1:\Delta\times\Omega^2\to\mathbb{R}^{n\times n}$$

are uniformly bounded, $\bar C_0(\cdot)$ is $\mathbb{F}$-progressively measurable, and for each $t\in[0,T]$, $s\mapsto\bar A_0(t,s)$ and $s\mapsto\bar A_1(t,s)$ are $\mathbb{F}$-progressively measurable and $\mathbb{F}^2$-progressively measurable on $[t,T]$, respectively.

We have the following result.

Theorem 5.8. *Let* (C3) *hold. In addition, suppose*

$$\begin{aligned}\bar A_0(t,s,\omega)\in\mathbb{M}_+^n,\quad\bar A_1(t,s,\omega,\omega')\in\widehat{\mathbb{M}}_+^n,\quad\bar C_0(s,\omega)\in\mathbb{M}_0^n,\\\text{a.e. }(t,s)\in\Delta^*,\ \text{a.s. }\omega,\omega'\in\Omega.\end{aligned}\tag{5.27}$$

Moreover, let $t\mapsto(\bar A_0(s,t),\bar C_0(s,t))$ be continuous, and

$$\begin{aligned}\bar A_0(s,t_1)^Tx\ge\bar A_0(s,t_0)^Tx,\quad\bar A_1(s,t_1)^Tx\ge\bar A_1(s,t_0)^Tx,\\\forall\,s\le t_0<t_1\le T,\ s\in[0,T],\ x\in\mathbb{R}_+^n,\ \text{a.s.}\end{aligned}\tag{5.28}$$

Let $(Y(\cdot), Z(\cdot\,,\cdot))$ be the adapted M-solution to (5.26) with $\psi(\cdot) \in L^2_{\mathcal{F}_T}(0,T;\mathbb{R}^n)$, $\psi(\cdot) \geq 0$. Then

$$\mathbb{E}\Big[\int_t^T Y(s)ds \mid \mathcal{F}_t\Big] \geq 0, \qquad \forall t \in [0,T], \text{ a.s.} \tag{5.29}$$

Proof. We consider the following linear MF-FSVIE:

$$\begin{aligned} X(t) = \varphi(t) + \int_0^t \Big(\bar{A}_0(s,t)^T X(s) + \mathbb{E}^*[\bar{A}_1(s,t)^T X(s)]\Big)ds \\ + \int_0^t \Big(\bar{C}_0(s)^T X(s)\Big)dW(s), \qquad t \in [0,T], \end{aligned} \tag{5.30}$$

where

$$\varphi(t) = \int_0^t \eta(s)ds, \qquad t \in [0,T],$$

for some $\eta(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ with $\eta(\cdot) \geq 0$. By our conditions on $\bar{A}_0(\cdot\,,\cdot)$ and $\bar{A}_1(\cdot\,,\cdot)$, using Proposition 5.6, we have

$$X(\cdot) \geq 0.$$

Then by Theorem 4.2, one obtains

$$\begin{aligned} 0 \leq \mathbb{E}\int_0^T \langle\, \psi(t), X(t)\,\rangle\, dt &= \mathbb{E}\int_0^T \langle\, \varphi(t), Y(t)\,\rangle\, dt \\ &= \mathbb{E}\int_0^T \int_0^t \langle\, \eta(s), Y(t)\,\rangle\, dsdt = \mathbb{E}\int_0^T \langle\, \eta(s), \int_s^T Y(t)dt\,\rangle\, ds. \end{aligned}$$

This proves (5.29). $\square$

Since the conditions assumed in Proposition 5.6 are very close to necessary conditions, we feel that it is very difficult (if not impossible) to get better comparison results for general MF-BSVIEs. However, if the drift term does not contain $Z(\cdot\,,\cdot)$, we are able to get a much better looking result. Let us now make it precise. For $i = 0, 1$, we consider the following (nonlinear) MF-BSVIEs:

$$Y_i(t) = \psi_i(t) + \int_t^T g_i(t,s,Y_i(s),\Gamma_i(t,s,Y_i(s)))ds - \int_t^T Z_i(t,s)dW(s), \quad t \in [0,T], \tag{5.31}$$

where

$$\begin{aligned} \Gamma_i(t,s,Y_i(s)) &= \mathbb{E}'\Big[\theta_i(t,s,Y_i(s),Y_i(s,\omega'))\Big] \\ &\equiv \int_\Omega \theta_i(t,s,\omega,\omega',Y_i(s,\omega),Y_i(s,\omega'))\mathbb{P}(d\omega'). \end{aligned} \tag{5.32}$$

Note that in the above, $Z_i(\cdot\,,\cdot)$ does not appear in the drift term.

Theorem 5.9. *Let $g_i : \Delta \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\theta_i : \Delta \times \Omega^2 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ satisfy* (H3)$_q$ *for some $q \geq 2$. Moreover, for all $y, y' \in \mathbb{R}^n$, $\gamma \in \mathbb{R}^m$, almost all $(t,s) \in \Delta$, and almost surely $\omega, \omega' \in \Omega$, the following hold:*

$$\left\{\begin{array}{ll} (g_0)_\gamma(t,s,\omega,y,\gamma) \in \widehat{\mathbb{M}}_+^{n\times m}, & (\theta_0)_{y'}(t,s,\omega,\omega',y,y') \in \widehat{\mathbb{M}}_+^{m\times n}, \\ (g_0)_y(t,s,\omega,y,\gamma) \in \widehat{M}_+^n, & (\theta_0)_y(t,s,\omega,\omega',y,y') \in \widehat{\mathbb{M}}_+^{m\times n}, \end{array}\right. \tag{5.33}$$

and

$$\begin{cases} g_1(t,s,\omega,y,\gamma) \geq g_0(t,s,\omega,y,\gamma), \\ \theta_1(t,s,\omega,\omega',y,y') \geq \theta_0(t,s,\omega,\omega',y,y'). \end{cases} \tag{5.34}$$

Let $\psi_i(\cdot) \in L^2_{\mathcal{F}_T}(0,T;\mathbb{R}^n)$ *with*

$$\psi_0(t) \leq \psi_1(t), \qquad \forall t \in [0,T], \text{ a.s. }, \tag{5.35}$$

and $(Y_i(\cdot), Z_i(\cdot\,,\cdot))$ *be the adapted M-solutions to the corresponding MF-BSVIEs* (5.31). *Then*

$$Y_0(t) \leq Y_1(t), \qquad t \in [0,T], \text{ a.s.} \tag{5.36}$$

Proof. From the MF-BSVIEs satisfied by $(Y_i(\cdot), Z_i(\cdot\,,\cdot))$, we have

$$\begin{aligned} Y_1(t) - Y_0(t) &= \psi_1(t) - \psi_0(t) + \int_t^T \Big[g_1(t,s,Y_1(s),\Gamma_1(t,s,Y_1(s))) \\ &\qquad - g_0(t,s,Y_0(s),\Gamma_0(t,s,Y_0(s))) \Big] ds - \int_t^T \Big[Z_1(t,s) - Z_0(t,s) \Big] dW(s) \\ &= \widehat{\psi}_1(t) - \widehat{\psi}_0(t) + \int_t^T \Big[g_0(t,s,Y_1(s),\Gamma_0(t,s,Y_1(s))) - g_0(t,s,Y_0(s),\Gamma_0(t,s,Y_0(s))) \Big] ds \\ &\qquad - \int_t^T \Big[Z_1(t,s) - Z_0(t,s) \Big] dW(s), \end{aligned}$$

where (making use of our condition)

$$\begin{aligned} &\widehat{\psi}_1(t) - \widehat{\psi}_0(t) = \psi_1(t) - \psi_0(t) + \int_t^T \Big(g_1(t,s,Y_1(s),\Gamma_1(t,s,Y_1(s))) - g_0(t,s,Y_1(s),\Gamma_0(t,s,Y_1(s))) \Big) ds \\ &= \psi_1(t) - \psi_0(t) + \int_t^T \Big(g_1(t,s,Y_1(s),\Gamma_1(t,s,Y_1(s))) - g_0(t,s,Y_1(s),\Gamma_1(t,s,Y_1(s))) \Big) ds \\ &\qquad + \int_t^T \Big(g_0(t,s,Y_1(s),\Gamma_1(t,s,Y_1(s))) - g_0(t,s,Y_1(s),\Gamma_0(t,s,Y_1(s))) \Big) ds \\ &= \psi_1(t) - \psi_0(t) + \int_t^T \Big(g_1(t,s,Y_1(s),\Gamma_1(t,s,Y_1(s))) - g_0(t,s,Y_1(s),\Gamma_1(t,s,Y_1(s))) \Big) ds \\ &\qquad + \int_t^T (\widetilde{g}_0)_\gamma(t,s) \Big(\Gamma_1(t,s,Y_1(s)) - \Gamma_0(t,s,Y_1(s)) \Big) ds \geq 0, \end{aligned}$$

with

$$\begin{aligned} &(\widetilde{g}_0)_\gamma(t,s) = \int_0^1 (g_0)_\gamma(t,s,Y_1(s),\Gamma_\lambda(t,s,Y_1(s))) d\lambda \in \widehat{\mathbb{M}}_+^{n\times m}, \\ &\Gamma_\lambda(t,s,Y_1(s)) = (1-\lambda)\Gamma_0(t,s,Y_1(s)) + \lambda \Gamma_1(t,s,Y_1(s)). \end{aligned}$$

Next, we note that

$$\begin{aligned} &g_0(t,s,Y_1(s),\Gamma_0(t,s,Y_1(s))) - g_0(t,s,Y_0(s),\Gamma_0(t,s,Y_0(s))) \\ &= \int_0^1 \Big\{ (g_0)_y(t,s,Y_\lambda(s),\Gamma_\lambda(t,s)) \Big[Y_1(s) - Y_0(s) \Big] \\ &\qquad + (g_0)_\gamma(t,s,Y_\lambda(s),\Gamma_\lambda(t,s)) \Big[\Gamma_0(t,s,Y_1(s)) - \Gamma_0(t,s,Y_0(s)) \Big] \Big\} d\lambda \\ &\equiv (g_0)_y(t,s) \Big[Y_1(s) - Y_0(s) \Big] + (g_0)_\gamma(t,s) \Big[\Gamma_0(t,s,Y_1(s)) - \Gamma_0(t,s,Y_0(s)) \Big], \end{aligned}$$

where

$$\begin{cases} Y_\lambda(s) = (1-\lambda)Y_0(s) + \lambda Y_1(s), \\ \Gamma_\lambda(t,s) = (1-\lambda)\Gamma_0(t,s,Y_0(s)) + \lambda\Gamma_0(t,s,Y_1(s)), \end{cases}$$

and

$$\begin{cases} (g_0)_y(t,s) = \displaystyle\int_0^1 (g_0)_y(t,s,Y_\lambda(s),\Gamma_\lambda(t,s))d\lambda \in \mathbb{M}_+^n, \\ (g_0)_\gamma(t,s) = \displaystyle\int_0^1 (g_0)_\gamma(t,s,Y_\lambda(s),\Gamma_\lambda(t,s))d\lambda \in \widehat{\mathbb{M}}_+^{n\times m}. \end{cases}$$

Also,

$$\begin{aligned} &\Gamma_0(t,s,Y_1(s)) - \Gamma_0(t,s,Y_0(s)) \\ &= \mathbb{E}'\Big[\theta_0(t,s,Y_1(s),Y_1(s,\omega')) - \theta_0(t,s,Y_0(s),Y_0(s,\omega'))\Big] \\ &= \mathbb{E}'\int_0^1 \Big\{(\theta_0)_y(t,s,Y_\lambda(s),Y_\lambda(s,\omega'))\Big(Y_1(s)-Y_0(s)\Big) \\ &\qquad\qquad +(\theta_0)_{y'}(t,s,Y_\lambda(s),Y_\lambda(s,\omega'))\Big(Y_1(s,\omega')-Y_0(s,\omega')\Big)\Big\}d\lambda \\ &\equiv \mathbb{E}'\Big[(\theta_0)_y(t,s)\Big]\Big(Y_1(s)-Y_0(s)\Big) + \mathbb{E}'\Big[(\theta_0)_{y'}(t,s)\Big(Y_1(s,\omega')-Y_0(s,\omega')\Big)\Big], \end{aligned}$$

with

$$\begin{cases} (\theta_0)_y(t,s) = \displaystyle\int_0^1 (\theta_0)_y(t,s,Y_\lambda(s),Y_\lambda(s,\omega'))d\lambda, \\ (\theta_0)_{y'}(t,s) = \displaystyle\int_0^1 (\theta_0)_{y'}(t,s,Y_\lambda(s),Y_\lambda(s,\omega'))d\lambda. \end{cases}$$

Thus,

$$\begin{aligned} Y_1(t) - Y_0(t) &= \widehat\psi_1(t) - \widehat\psi_0(t) + \int_t^T \Big\{\bar A_0(t,s)\Big(Y_1(s)-Y_0(s)\Big) \\ &\quad +\mathbb{E}'\Big[\bar A_1(t,s)\Big(Y_1(s)-Y_0(s)\Big)\Big]\Big\}ds - \int_t^T \Big(Z_1(t,s)-Z_0(s,t)\Big)dW(s), \quad t\in[0,T], \end{aligned} \tag{5.37}$$

with

$$\begin{cases} \bar A_0(t,s) = (g_0)_y(t,s) + \mathbb{E}'\Big[(g_0)_\gamma(t,s)(\theta_0)_y(t,s)\Big] \in \widehat{\mathbb{M}}_+^n, \\ \bar A_1(t,s) = (g_0)_\gamma(t,s)(\theta_0)_{y'}(t,s) \in \widehat{\mathbb{M}}_+^n, \end{cases} \qquad (t,s)\in\Delta, \text{ a.s.} \tag{5.38}$$

Now, for any $\varphi(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, let $X(\cdot)$ be the solution to the following linear MF-FSVIE:

$$X(t) = \varphi(t) + \int_0^t \Big(\bar A_0(s,t)^T X(s) + \mathbb{E}^*\Big[\bar A_1(s,t)^T X(s)\Big]\Big)ds, \qquad t\in[0,T]. \tag{5.39}$$

By Proposition 5.5, we know that $X(\cdot)\ge 0$. Then by Theorem 4.2, we have

$$0 \le \mathbb{E}\int_0^T \langle\, \widehat\psi_1(t) - \widehat\psi_0(t), X(t)\,\rangle\, dt = \mathbb{E}\int_0^T \langle\, \varphi(t), Y_1(t)-Y_0(t)\,\rangle\, dt.$$

Hence, (5.36) follows. □

Combining the above two results, we are able to get a comparison theorem for the following MF-BSVIE:

$$\begin{aligned} Y_i(t) = \psi_i(t) + \int_t^T \Big(g_i(t,s,Y_i(s),\Gamma_i(t,s,Y_i(s))) + \bar{C}_0(t)Z_i(s,t)\Big)ds \\ - \int_t^T Z_i(t,s)dW(s), \qquad t \in [0,T], \end{aligned} \tag{5.40}$$

where $\Gamma_i(\cdot)$ is as that in (5.31). Under proper conditions, we will have the following comparison:

$$\mathbb{E}\Big[\int_t^T Y_0(s)ds \mid \mathcal{F}_t\Big] \le \mathbb{E}\Big[\int_t^T Y_1(s)ds \mid \mathcal{F}_t\Big], \qquad \forall t \in [0,T], \text{ a.s.} \tag{5.41}$$

We omit the details here.

We note that in Proposition 5.6, monotonicity conditions for $\varphi(\cdot)$, $A_0(\cdot\,,\cdot)$ and $C_0(\cdot\,,\cdot)$ play a crucial role. These kind of conditions were overlooked in [39, 40, 41]. The following example shows that in general (5.36) might be false.

Example 5.10. Consider

$$Y_0(t) = -\int_t^T Y_0(s)ds, \qquad t \in [0,T],$$

and

$$Y_1(t) = t - \int_t^T Y_1(s)ds, \qquad t \in [0,T].$$

Then

$$Y_0(t) = 0, \qquad t \in [0,T],$$

and the equation for $Y_1(\cdot)$ is equivalent to the following:

$$\dot{Y}_1(t) = Y_1(t) + 1, \qquad Y_1(T) = T,$$

whose solution is given by

$$Y_1(t) = e^{t-T}(T+1) - 1, \qquad t \in [0,T].$$

It is easy to see that

$$Y_1(t) < 0 = Y_0(t), \qquad \forall t \in [0, T - \ln(T+1)).$$

Hence, (5.36) fails.

To conclude this section, we would like to pose the following open question: For general MF-BSVIEs, under what conditions on the coefficients, one has a nice-looking comparison theorem?

We hope to be able to report some results concerning the above question before long.

6 An Optimal Control Problem for MF-SVIEs.

In this section, we will briefly discuss a simple optimal control problem for MF-FSVIEs. This can be regarded as an application of Theorem 4.1, a duality principle for MF-FSVIEs. The main clue is similar to the relevant results presented in [39, 41]. We will omit some detailed derivations. General optimal control problems for MF-FSVIEs will be much more involved and we will present systematic results for that in our forthcoming publications.

Let U be a non-empty bounded convex set in $\mathbb{R}^m$, and let $\mathcal{U}$ be the set of all $\mathbb{F}$-adapted processes $u : [0,T]\times\Omega\to U$. Since U is bounded, we see that $\mathcal{U}\subseteq L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^m)$. For any $u(\cdot)\in\mathcal{U}$, consider the following controlled MF-FSVIE:

$$\begin{aligned}X(t)=\varphi(t)&+\int_0^t b(t,s,X(s),u(s),\Gamma^b(t,s,X(s),u(s)))ds\\&+\int_0^t\sigma(t,s,X(s),u(s),\Gamma^\sigma(t,s,X(s),u(s)))dW(s),\qquad t\in[0,T],\end{aligned}\tag{6.1}$$

where

$$\left\{\begin{array}{l}b:\Delta^*\times\Omega\times\mathbb{R}^n\times U\times\mathbb{R}^{m_1}\to\mathbb{R}^n,\\ \sigma:\Delta^*\times\Omega\times\mathbb{R}^n\times U\times\mathbb{R}^{m_2}\to\mathbb{R}^n,\end{array}\right.$$

and

$$\left\{\begin{aligned}\Gamma^b(t,s,X(s),u(s))&=\int_\Omega\theta^b(t,s,\omega,\omega',X(s,\omega),u(s,\omega),X(s,\omega'),u(s,\omega'))\mathbb{P}(d\omega')\\&\equiv\mathbb{E}'\Big[\theta^b(t,s,X(s),u(s),x',u')\Big]_{(x',u')=(X(s),u(s))},\\ \Gamma^\sigma(t,s,X(s),u(s))&=\int_\Omega\theta^\sigma(t,s,\omega,\omega',X(s,\omega),u(s,\omega),X(s,\omega'),u(s,\omega'))\mathbb{P}(d\omega')\\&\equiv\mathbb{E}'\Big[\theta^\sigma(t,s,X(s),u(s),x',u')\Big]_{(x',u')=(X(s),u(s))},\end{aligned}\right.$$

with

$$\left\{\begin{array}{l}\theta^b:\Delta^*\times\Omega^2\times\mathbb{R}^n\times U\times\mathbb{R}^n\times U\to\mathbb{R}^{m_1},\\ \theta^\sigma:\Delta^*\times\Omega^2\times\mathbb{R}^n\times U\times\mathbb{R}^n\times U\to\mathbb{R}^{m_2}.\end{array}\right.$$

In the above, $X(\cdot)$ is referred to as the *state process* and $u(\cdot)$ as the *control process*. We introduce the following assumptions for the state equation (Comparing with (H1)–(H2)):

(H1)$''$ The maps

$$\left\{\begin{array}{l}b:\Delta^*\times\Omega\times\mathbb{R}^n\times U\times\mathbb{R}^{m_1}\to\mathbb{R}^n,\\ \sigma:\Delta^*\times\Omega\times\mathbb{R}^n\times U\times\mathbb{R}^{m_2}\to\mathbb{R}^n,\end{array}\right.$$

are measurable, and for all $(t,x,u,\gamma,\gamma')\in[0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}$, the map

$$(s,\omega)\mapsto(b(t,s,\omega,x,u,\gamma),\sigma(t,s,\omega,x,u,\gamma'))$$

is $\mathbb{F}$-progressively measurable on $[0,t]$. Moreover, for all $(t,s,\omega,\omega')\in\Delta^c\times\Omega$, the map

$$(x,u,\gamma,\gamma')\mapsto(b(t,s,\omega,x,u,\gamma),\sigma(t,s,x,u,\gamma'))$$

is continuously differentiable and there exists some constant $L>0$ such that

$$\begin{aligned}&|b_x(t,s,\omega,x,u,\gamma)|+|b_u(t,s,\omega,x,u,\gamma)|+|b_\gamma(t,s,\omega,x,u,\gamma)|\\&+|\sigma_x(t,s,\omega,x,u,\gamma')|+|\sigma_u(t,s,\omega,x,u,\gamma')|+|\sigma_{\gamma'}(t,s,\omega,x,u,\gamma')|\le L,\\&\qquad(t,s,\omega,x,u,\gamma,\gamma')\in\Delta^*\times\Omega\times\mathbb{R}^n\times U\times\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}.\end{aligned}\tag{6.2}$$

Further,

$$\begin{aligned}&|b(t,s,\omega,x,u,\gamma)|+|\sigma(t,s,\omega,x,u,\gamma')|\le L(1+|x|+|\gamma|+|\gamma'|),\\&\qquad(t,s,\omega,x,u,\gamma,\gamma')\in\Delta^*\times\Omega\times\mathbb{R}^n\times U\times\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}.\end{aligned}\tag{6.3}$$

(H2)$''$ The maps

$$\begin{cases}\theta^b:\Delta^*\times\Omega^2\times\mathbb{R}^n\times U\times\mathbb{R}^n\times U\to\mathbb{R}^{m_1},\\ \theta^\sigma:\Delta^*\times\Omega^2\times\mathbb{R}^n\times U\times\mathbb{R}^n\times U\to\mathbb{R}^{m_2},\end{cases}$$

are measurable, and for all $(t,x,u,x',u')\in[0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^n\times U$, the map

$$(s,\omega,\omega')\mapsto(\theta^b(t,s,\omega,\omega',x,u,x',u'),\theta^\sigma(t,s,\omega,\omega',x,u,x',u'))$$

is $\mathbb{F}^2$-progressively measurable on $[0,t]$. Moreover, for any $(t,s,\omega,\omega')\in\Delta^*\times\Omega^2$,

$$(x,u,\gamma,\gamma')\mapsto(\theta^b(t,s,\omega,\omega',x,u,x',u'),\theta^\sigma(t,s,\omega,\omega',x,u,x',u'))$$

is continuously differentiable and there exists some constant $L>0$ such that

$$\begin{aligned}&|\theta^b_x(t,s,\omega,\omega',x,u,x',u')|+|\theta^b_u(t,s,\omega,\omega',x,u,x',u')|\\&\quad+|\theta^b_{x'}(t,s,\omega,\omega',x,u,x',u')|+|\theta^b_{u'}(t,s,\omega,\omega',x,u,x',u')|\\&\quad+|\theta^\sigma_x(t,s,\omega,\omega',x,u,x',u')|+|\theta^\sigma_u(t,s,\omega,\omega',x,u,x',u')|\\&\quad+|\theta^\sigma_{x'}(t,s,\omega,\omega',x,u,x',u')|+|\theta^\sigma_{u'}(t,s,\omega,\omega',x,u,x',u')|\le L,\\&\qquad(t,s,\omega,\omega',x,u,x',u')\in\Delta^*\times\Omega^2\times\mathbb{R}^n\times U\times\mathbb{R}^n\times U.\end{aligned}\tag{6.4}$$

Further,

$$\begin{aligned}&|\theta^b(t,s,\omega,\omega',x,u,x',u')|+|\theta^\sigma(t,s,\omega,\omega',x,u,x',u')|\le L(1+|x|+|x'|),\\&\qquad(t,s,\omega,\omega',x,u,x',u')\in\Delta^*\times\Omega^2\times\mathbb{R}^n\times U\times\mathbb{R}^n\times U.\end{aligned}\tag{6.5}$$

It is easy to see that under (H1)$''$–(H2)$''$, for any given $u(\cdot)\in\mathcal{U}$, the state equation (6.1) satisfies (H1)–(H2). Hence, for any $\varphi(\cdot)\in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$, (6.1) admits a unique solution $X(\cdot)\in L^p(0,T;\mathbb{R}^n)$.

To measure the performance of the control process $u(\cdot)$, the following (Lagrange type) *cost functional* is defined:

$$J(u(\cdot))=\mathbb{E}\int_0^T g(s,X(s),u(s),\Gamma^g(s,X(s),u(s)))ds,\tag{6.6}$$

where

$$g:[0,T]\times\Omega\times\mathbb{R}^n\times U\times\mathbb{R}^\ell\to\mathbb{R},$$

and

$$
\begin{aligned}
\Gamma^g(s,X(s),u(s)) &= \int_\Omega \theta^g(s,\omega,\omega',X(s,\omega),u(s,\omega),X(s,\omega'),u(s,\omega'))\mathbb{P}(d\omega') \\
&\equiv \mathbb{E}'\Big[\theta^g(s,X(s),u(s),x',u')\Big]_{(x',u')=(X(s),u(s))},
\end{aligned}
$$

with

$$
\theta^g : [0,T]\times\Omega^2\times\mathbb{R}^n\times U\times\mathbb{R}^n\times U \to \mathbb{R}^\ell .
$$

For convenience, we make the following assumptions for the functions involved in the cost functional.

(H1)‴ The map $g : [0,T]\times\Omega\times\mathbb{R}^n\times U\times\mathbb{R}^\ell \to \mathbb{R}$ is measurable, and for all $(x,u,\gamma)\in\mathbb{R}^n\times U\times\mathbb{R}^\ell$, the map $(t,\omega)\mapsto g(t,\omega,x,u,\gamma)$ is $\mathbb{F}$-progressively measurable. Moreover, for almost all $(t,\omega)\in\Delta^*\times\Omega$, the map $(x,u,\gamma)\mapsto g(t,\omega,x,u,\gamma)$ is continuously differentiable and there exists some constant $L>0$ such that

$$
\begin{aligned}
&|g_x(t,\omega,x,u,\gamma)| + |g_u(t,\omega,x,u,\gamma)| + |g_\gamma(t,\omega,x,u,\gamma)| \le L, \\
&\qquad (t,\omega,x,u,\gamma)\in[0,T]\times\Omega\times\mathbb{R}^n\times U\times\mathbb{R}^\ell .
\end{aligned}
\tag{6.7}
$$

Further,

$$
\begin{aligned}
&|g(t,\omega,x,u,\gamma)| \le L(1+|x|+|\gamma|), \\
&\qquad\qquad (t,\omega,x,u,\gamma)\in[0,T]\times\Omega\times\mathbb{R}^n\times U\times\mathbb{R}^\ell .
\end{aligned}
\tag{6.8}
$$

(H2)‴ The map $\theta^g : [0,T]\times\Omega^2\times\mathbb{R}^n\times U\times\mathbb{R}^n\times U \to \mathbb{R}^\ell$ is measurable, and for all $(x,u,x',u')\in\mathbb{R}^n\times U\times\mathbb{R}^n\times U$, the map $(s,\omega,\omega')\mapsto(\theta^g(t,\omega,\omega',x,u,x',u')$ is $\mathbb{F}^2$-progressively measurable. Moreover, for almost all $(t,\omega,\omega')\in[0,T]\times\Omega^2$, the map $(x,u,x',u')\mapsto\theta^g(t,s,\omega,\omega',x,u,x',u')$ is continuously differentiable and there exists some constant $L>0$ such that

$$
\begin{aligned}
&|\theta^g_x(t,\omega,\omega',x,u,x',u')| + |\theta^g_u(t,\omega,\omega',x,u,x',u')| \\
&\quad + |\theta^g_{x'}(t,\omega,\omega',x,u,x',u')| + |\theta^g_{u'}(t,\omega,\omega',x,u,x',u')| \le L, \\
&\qquad (t,\omega,\omega',x,u,x',u')\in[0,T]\times\Omega^2\times\mathbb{R}^n\times U\times\mathbb{R}^n\times U .
\end{aligned}
\tag{6.9}
$$

Further,

$$
\begin{aligned}
&|\theta^g(t,\omega,\omega',x,u,x',u')| \le L(1+|x|+|x'|), \\
&\qquad (t,\omega,\omega',x,u,x',u')\in[0,T]\times\Omega^2\times\mathbb{R}^n\times U\times\mathbb{R}^n\times U .
\end{aligned}
\tag{6.10}
$$

Under (H1)″–(H2)″ and (H1)‴–(H2)‴, the cost functional $J(u(\cdot))$ is well-defined. Then we can state our optimal control problem as follows.

Problem (C). For given $\varphi(\cdot)\in L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$, find $\bar u(\cdot)\in\mathcal{U}$ such that

$$
J(\bar u(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}} J(u(\cdot)). \tag{6.11}
$$

Any $\bar u(\cdot)\in\mathcal{U}$ satisfying (6.11) is called an *optimal control* of Problem (C), and the corresponding state process $\bar X(\cdot)$ is called an *optimal state process*. In this case, we refer to $(\bar X(\cdot),\bar u(\cdot))$ as an *optimal pair*.

We now briefly derive the Pontryagin type maximum principle for any optimal pair $(\bar X(\cdot),\bar u(\cdot))$. To this end, we take any $u(\cdot)\in\mathcal{U}$, let

$$
u^\varepsilon(\cdot) = \bar u(\cdot) + \varepsilon[u(\cdot)-\bar u(\cdot)] = (1-\varepsilon)\bar u(\cdot) + \varepsilon u(\cdot) \in \mathcal{U}.
$$

Let $X^\varepsilon(\cdot)$ be the corresponding state process. Then

$$X_1(\cdot) = \lim_{\varepsilon \to 0} \frac{X^\varepsilon(\cdot) - \bar{X}(\cdot)}{\varepsilon}$$

satisfies the following:

$$\begin{aligned}
X_1(t) = & \int_0^t \Big\{ b_x(t,s)X_1(s) + b_u(t,s)[u(s) - \bar{u}(s)] \\
& \qquad + b_\gamma(t,s)\mathbb{E}'\Big[\theta_x^b(t,s)X_1(s,\omega) + \theta_u^b(t,s)[u(s,\omega) - \bar{u}(s,\omega)] \\
& \qquad + \theta_{x'}^b(t,s)X_1(s,\omega') + \theta_{u'}^b(t,s)[u(s,\omega') - \bar{u}(s,\omega')]\Big]\Big\}ds \\
& + \int_0^t \Big\{ \sigma_x(t,s)X_1(s) + \sigma_u(t,s)[u(s) - \bar{u}(s)] \\
& \qquad + \sigma_\gamma(t,s)\mathbb{E}'\Big[\theta_x^\sigma(t,s)X_1(s,\omega) + \theta_u^\sigma(t,s)[u(s,\omega) - \bar{u}(s,\omega)] \\
& \qquad + \theta_{x'}^\sigma(t,s)X_1(s,\omega') + \theta_{u'}^\sigma(t,s)[u(s,\omega') - \bar{u}(s,\omega')]\Big]\Big\}dW(s) \\
= & \int_0^t \Big\{ \Big[b_x(t,s) + b_\gamma(t,s)\mathbb{E}'\theta_x^b(t,s)\Big]X_1(s) \\
& \qquad + \Big[b_u(t,s) + b_\gamma(t,s)\mathbb{E}'\theta_u^b(t,s)\Big][u(s) - \bar{u}(s)] \\
& \qquad + \mathbb{E}'\Big[b_\gamma(t,s)\theta_{x'}^b(t,s)X_1(s) + b_\gamma(t,s)\theta_{u'}^b(t,s)[u(s) - \bar{u}(s)]\Big]\Big\}ds \\
& + \int_0^t \Big\{ \Big[\sigma_x(t,s) + \sigma_\gamma(t,s)\mathbb{E}'\theta_x^\sigma(t,s)\Big]X_1(s) \\
& \qquad + \Big[\sigma_u(t,s) + \sigma_\gamma(t,s)\mathbb{E}'\theta_u^\sigma(t,s)\Big][u(s) - \bar{u}(s)] \\
& \qquad + \mathbb{E}'\Big[\sigma_\gamma(t,s)\theta_{x'}^\sigma(t,s)X_1(s) + \sigma_\gamma(t,s)\theta_{u'}^\sigma(t,s)[u(s) - \bar{u}(s)]\Big]\Big\}dW(s) \\
\equiv & \int_0^t \Big\{ A_0(t,s)X_1(s) + B_0(t,s)[u(s) - \bar{u}(s)] + \mathbb{E}'\Big[C_0(t,s)X_1(s) + D_0(t,s)[u(s) - \bar{u}(s)]\Big]\Big\}ds \\
& + \int_0^t \Big\{ A_1(t,s)X_1(s) + B_1(t,s)[u(s) - \bar{u}(s)] + \mathbb{E}'\Big[C_1(t,s)X_1(s) + D_1(t,s)[u(s) - \bar{u}(s)]\Big]\Big\}dW(s) \\
\equiv & \,\widehat{\varphi}(t) + \int_0^t \Big\{ A_0(t,s)X_1(s) + \mathbb{E}'\Big[C_0(t,s)X_1(s)\Big]\Big\}ds \\
& \qquad + \int_0^t \Big\{ A_1(t,s)X_1(s) + \mathbb{E}'\Big[C_1(t,s)X_1(s)\Big]\Big\}dW(s),
\end{aligned}$$

where

$$\begin{cases}
b_\xi(t,s) = b_\xi(t,s,\bar{X}(s),\bar{u}(s),\Gamma^b(t,s,\bar{X}(s),\bar{u}(s))), \qquad \xi = x,u,\gamma, \\
\theta_\xi^b(t,s) = \theta_\xi^b(t,s,\omega,\omega',\bar{X}(s,\omega),\bar{u}(s,\omega),\bar{X}(s,\omega'),\bar{u}(s,\omega')), \qquad \xi = x,u,x',u', \\
\sigma_\xi(t,s) = \sigma_\xi(t,s,\bar{X}(s),\bar{u}(s),\Gamma^\sigma(t,s,\bar{X}(s),\bar{u}(s))), \qquad \xi = x,u,\gamma, \\
\theta_\xi^\sigma(t,s) = \theta_\xi^b(t,s,\omega,\omega',\bar{X}(s,\omega),\bar{u}(s,\omega),\bar{X}(s,\omega'),\bar{u}(s,\omega')), \qquad \xi = x,u,x',u',
\end{cases}$$

and

$$\begin{cases}
A_0(t,s) = b_x(t,s) + b_\gamma(t,s)\mathbb{E}'\theta_x^b(t,s), \qquad B_0(t,s) = b_u(t,s) + b_\gamma(t,s)\mathbb{E}'\theta_u^b(t,s), \\
C_0(t,s) = b_\gamma(t,s)\theta_{x'}^b(t,s), \qquad D_0(t,s) = b_\gamma(t,s)\theta_{u'}^b(t,s), \\
A_1(t,s) = \sigma_x(t,s) + \sigma_\gamma(t,s)\mathbb{E}'\theta_x^\sigma(t,s), \qquad B_1(t,s) = \sigma_u(t,s) + \sigma_\gamma(t,s)\mathbb{E}'\theta_u^\sigma(t,s), \\
C_1(t,s) = \sigma_\gamma(t,s)\theta_{x'}^\sigma(t,s), \qquad D_1(t,s) = \sigma_\gamma(t,s)\theta_{u'}^\sigma(t,s).
\end{cases}$$

Also,

$$\widehat{\varphi}(t)=\int_0^t\Big\{B_0(t,s)[u(s)-\bar u(s)]+\mathbb{E}'\Big[D_0(t,s)[u(s)-\bar u(s)]\Big]\Big\}ds$$
$$+\int_0^t\Big\{B_1(t,s)[u(s)-\bar u(s)]+\mathbb{E}'\Big[D_1(t,s)[u(s)-\bar u(s)]\Big]\Big\}dW(s).$$

On the other hand, by the optimality of $(\bar X(\cdot),\bar u(\cdot))$, we have

$$\begin{aligned}
0&\le\lim_{\varepsilon\to0}\frac{J(u^\varepsilon(\cdot))-J(\bar u(\cdot))}{\varepsilon}\\
&=\mathbb{E}\int_0^T\Big\{g_x(s)X_1(s)+g_u(s)[u(s)-\bar u(s)]\\
&\qquad+g_\gamma(s)\mathbb{E}'\Big[\theta^g_x(s)X_1(s,\omega)+\theta^g_u(s)[u(s,\omega)-\bar u(s,\omega)]\\
&\qquad+\theta^g_{x'}(s)X_1(s,\omega')+\theta^g_{u'}(s)[u(s,\omega')-\bar u(s,\omega')]\Big]\Big\}ds\\
&=\mathbb{E}\int_0^T\Big\{\Big[g_x(s)+g_\gamma(s)\mathbb{E}'\theta^g_x(s)\Big]X_1(s)+\Big[g_u(s)+g_\gamma(s)\mathbb{E}'\theta^g_u(s)\Big][u(s)-\bar u(s)]\\
&\qquad+\mathbb{E}'\Big[g_\gamma(s)\theta^g_{x'}(s)X_1(s)+g_\gamma(s)\theta^g_{u'}(s)[u(s)-\bar u(s)]\Big]\Big\}ds\\
&=\mathbb{E}\int_0^T\Big\{a_0(s)^TX_1(s)+b_0(s)^T[u(s)-\bar u(s)]\\
&\qquad+\mathbb{E}'\Big[c_0(s)^TX_1(s)+d_0(s)^T[u(s)-\bar u(s)]\Big]\Big\}ds\\
&=\mathbb{E}\Big\{\widehat{\varphi}_0+\int_0^T\Big(a_0(s)^TX_1(s)+\mathbb{E}'\Big[c_0(s)^TX_1(s)\Big]\Big)ds\Big\},
\end{aligned}$$

where

$$\begin{cases}g_\xi(s)=g_\xi(s,\bar X(s),\bar u(s),\Gamma^g(s,\bar X(s),\bar u(s))),\qquad \xi=x,u,\gamma,\\ \theta^g_\xi(s)=\theta^g_\xi(s,\omega,\omega',\bar X(s,\omega),\bar u(s,\omega),\bar X(s,\omega'),\bar u(s,\omega')),\qquad \xi=x,u,x',u',\end{cases}$$

and

$$\begin{cases}a_0(s)^T=g_x(s)+g_\gamma(s)\mathbb{E}'\theta^g_x(s),\qquad b_0(s)^T=g_u(s)+g_\gamma(s)\mathbb{E}'\theta^g_u(s),\\ c_0(s)^T=g_\gamma(s)\theta^g_{x'}(s),\qquad d_0(s)^T=g_\gamma(s)\theta^g_{u'}(s),\\ \widehat{\varphi}_0=\displaystyle\int_0^T\Big\{b_0(s)^T[u(s)-\bar u(s)]+\mathbb{E}'\Big[d_0(s)^T[u(s)-\bar u(s)]\Big]\Big\}ds.\end{cases}$$

Then for any undetermined $(Y(\cdot),Z(\cdot\,,\cdot))\in\mathcal{M}^2[0,T]$, similar to the proof of Theorem 4.1, we have

$$\mathbb{E}\int_0^T\langle\,Y(t),\widehat{\varphi}(t)\,\rangle\,dt=\mathbb{E}\int_0^T\langle\,X_1(t),Y(t)-\int_t^T\Big(A_0(s,t)^TY(s)+A_1(s,t)^TZ(s,t)$$
$$+\mathbb{E}^*\Big[C_0(s,t)^TY(s)+C_1(s,t)^TZ(s,t)\Big]\Big)ds\,\rangle\,dt.$$

Hence,

$$
\begin{aligned}
0 &\leq \mathbb{E}\Big\{\widehat{\varphi}_0+\int_0^T\Big(a_0(s)^TX_1(s)+\mathbb{E}'\Big[c_0(s)^TX_1(s)\Big]\Big)ds\Big\}\\
&=\mathbb{E}\Big\{\widehat{\varphi}_0-\int_0^T\langle\, Y(t),\widehat{\varphi}(t)\,\rangle\,dt+\int_0^T\langle\, X_1(t),Y(t)-\int_t^T\Big(A_0(s,t)^TY(s)\\
&\qquad\qquad +A_1(s,t)^TZ(s,t)+\mathbb{E}^*\Big[C_0(s,t)^TY(s)+C_1(s,t)^TZ(s,t)\Big]\Big)ds\,\rangle\,dt\\
&\qquad\qquad +\int_0^T\Big(\langle\, X_1(t),a_0(t)\,\rangle+\mathbb{E}'\Big[\langle\, X_1(t),c_0(t)\,\rangle\Big]\Big)dt\Big\}\\
&=\mathbb{E}\Big\{\widehat{\varphi}_0-\int_0^T\langle\, Y(t),\widehat{\varphi}(t)\,\rangle\,dt+\int_0^T\langle\, X_1(t),Y(t)+a_0(t)+\mathbb{E}^*c_0(t)\\
&\qquad -\int_t^T\Big(A_0(s,t)^TY(s)+A_1(s,t)^TZ(s,t)\\
&\qquad\qquad +\mathbb{E}^*\Big[C_0(s,t)^TY(s)+C_1(s,t)^TZ(s,t)\Big]\Big)ds\,\rangle\,dt\Big\}.
\end{aligned}
$$

We now let $(Y(\cdot),Z(\cdot\,,\cdot))\in\mathcal{M}^2[0,T]$ be the adapted M-solution to the following MF-BSVIE:

$$
\begin{aligned}
Y(t)=-a_0(t)-\mathbb{E}^*c_0(t)&+\int_t^T\Big(A_0(s,t)^TY(s)+A_1(s,t)^TZ(s,t)\\
&+\mathbb{E}^*\Big[C_0(s,t)^TY(s)+C_1(s,t)^TZ(s,t)\Big]\Big)dsdt-\int_t^TZ(t,s)dW(s).
\end{aligned}\tag{6.12}
$$

Then

$$
\begin{aligned}
0&\leq\mathbb{E}\Big\{\widehat{\varphi}_0-\int_0^T\langle\, Y(t),\widehat{\varphi}(t)\,\rangle\,dt\Big\}\\
&=\mathbb{E}\Big\{\int_0^T\Big\{\langle\, b_0(t),u(t)-\bar{u}(t)\,\rangle+\mathbb{E}'\Big[\langle\, d_0(t),u(t)-\bar{u}(t)\,\rangle\Big]\Big\}dt\\
&\qquad-\int_0^T\langle\, Y(t),\int_0^t\Big(B_0(t,s)[u(s)-\bar{u}(s)]+\mathbb{E}'\Big[D_0(t,s)[u(s)-\bar{u}(s)]\Big]\Big)ds\\
&\qquad\qquad+\int_0^t\Big(B_1(t,s)[u(s)-\bar{u}(s)]+\mathbb{E}'\Big[D_1(t,s)[u(s)-\bar{u}(s)]\Big]\Big)dW(s)\,\rangle\,dt\Big\}\\
&=\mathbb{E}\Big\{\int_0^T\Big(\langle\, b_0(t)+[\mathbb{E}^*d_0(t)],u(t)-\bar{u}(t)\,\rangle\Big)dt\\
&\qquad-\int_0^T\langle\int_t^T\Big(B_0(s,t)^TY(s)+\mathbb{E}^*[D_0(s,t)^TY(s)]\Big)ds,u(t)-\bar{u}(t)\,\rangle\,dt\\
&\qquad-\int_0^T\langle\int_t^T\Big(B_1(s,t)^TZ(s,t)+\mathbb{E}^*[D_1(s,t)^TZ(s,t)]\Big)ds,u(t)-\bar{u}(t)\,\rangle\,dt\Big\}.
\end{aligned}
$$

Hence, we must have the following variational inequality:

$$
\begin{aligned}
\langle\, b_0(t)+[\mathbb{E}^*d_0(t)]&-\int_t^T\Big(B_0(s,t)^TY(s)+\mathbb{E}^*[D_0(s,t)^TY(s)]\\
&+B_1(s,t)^TZ(s,t)+\mathbb{E}^*[D_1(s,t)^TZ(s,t)]\Big)ds,u-\bar{u}(t)\,\rangle\geq 0,\\
&\qquad\qquad\qquad\forall u\in U,\ \text{a.e. } t\in[0,T],\ \text{a.s.}
\end{aligned}\tag{6.13}
$$

We now summarize the above derivation.

Theorem 5.1. *Let* (H1)$''$–(H2)$''$ *and* (H1)$'''$–(H2)$'''$ *hold and let* $(\bar{X}(\cdot),\bar{u}(\cdot))$ *be an optimal pair of Problem* (C)*. Then the adjoint equation* (6.12) *admits a unique adapted M-solution* $(Y(\cdot),Z(\cdot\,,\cdot))\in\mathcal{M}^2[0,T]$ *such that the variational inequality* (6.13) *holds.*

The purpose of presenting a simple optimal control problem of MF-FSVIEs here is to realize a major motivation of studying MF-BSVIEs. It is possible to discuss Bolza type cost functional. Also, some of the assumptions assumed in this section might be relaxed. However, we have no intention to have a full exploration of general optimal control problems for MF-FSVIEs in the current paper since such kind of general problems (even for FSVIEs) are much more involved and they deserve to be addressed in another paper. We will report further results along that line in a forthcoming paper.

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